AWS 2025 extended lecture notes
Topic: Representations of p-adic groups

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Abstract

This is an expanded version of the topics covered during the Arizona Winter School mini-course.

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Contents

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Source of some of the material: Section 1.2 is taken from [Fin23] and Section 2 and Section 3 are taken from [Fin], with some modifications.

1 p-adic groups and their representations

1.1 The set-up

Through these notes, we let F be a non-archimedean local field with residue field $\mathbb{F}_q$. The discrete valuation of F endows F with a topology. The field $\mathbb{F}_q$ has cardinality q and characteristic p. Let G be a connected reductive group over F. These groups include $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}, \ldots$, but also groups of exceptional types G_2, F_4, E_6, E_7 and E_8. We assume that the reader has a basic familiarity with reductive groups. For a brief introduction to reductive groups, see [Fin23, Sections 2.1 and 2.3]. In this lecture series we are interested in the topological group $G(F)$ to which we also refer as a *p-adic group*, where the topology in $G(F)$ is the one coming from the topolgy on F. For example, a basis of open neighborhoods of the identity 1 in $\mathrm{GL}_n(F)$ consists of the subgroups

$$1 + \varpi \,\mathrm{Mat}_{n\times n}(\mathcal{O}_F) \supset 1 + \varpi^2 \,\mathrm{Mat}_{n\times n}(\mathcal{O}_F) \supset 1 + \varpi^3 \,\mathrm{Mat}_{n\times n}(\mathcal{O}_F) \supset \ldots,$$

where ϖ denotes a uniformizer of F.

1.2 Representations of p-adic groups

Definition 1.2.1. A *smooth representation* (π, V) of $G(F)$ is

- a complex vector space V and
- a group homomorphism $\pi : G(F) \to \mathrm{Aut}(V)$

such that for every $v \in V$ the stabilizer $\mathrm{Stab}(v) = \{g \in G(F) \,|\, \pi(g)v = v\}$ of v in $G(F)$ is an open subset of $G(F)$.

We define smooth representations of closed subgroups of $G(F)$ (with respect to the p-adic topology underlying the topological group $G(F)$) analogously.

In this survey we will focus on the *irreducible* smooth representations, i.e. those smooth representations (π, V) that have precisely two subrepresentations (subspaces of V preserved under the action of $G(F)$): the trivial representation on the zero dimensional vector space $\{0\}$ and the representation (π, V) itself.

An important finiteness property of smooth representations is the following.

Definition 1.2.2. A smooth representation (π, V) of $G(F)$ is called *admissible* if the space

$$V^K := \{v \in V \,|\, \pi(k)v = v \;\forall\, k \in K\}$$

of K-fixed vectors has finite dimension for every compact open subgroup K of $G(F)$.

An important fact for representations with complex coefficients is that irreducible smooth representations are automatically admissible.

Fact 1.2.3. *If (π, V) is an irreducible smooth representation of $G(F)$, then (π, V) is admissible.*

An important tool to construct representations is the induction. There are two kinds of inductions that will play an important role for us.

Definition 1.2.4. Let H be a closed subgroup of $G(F)$ (with respect to the p-adic topology underlying the topological group $G(F)$) and let (σ, W) be a smooth representation of H.

The *induced representation* $(R, \mathrm{Ind}_H^{G(F)} W)$ (also sometimes referred to as *smooth induction*) is defined as follows:

- $\mathrm{Ind}_H^{G(F)} W$ is the space of functions $f : G(F) \to W$ satisfying
  (a) $f(hg) = \sigma(h)f(g)$ for all $h \in H, g \in G(F)$, and
  (b) there exists a compact open subgroup $K_f \subset G(F)$ such that $f(gk) = f(g)$ for all $k \in K_f$.
- The action of $G(F)$ on $\mathrm{Ind}_H^{G(F)} W$ is via right translation, i.e.

$$(R(g)(f))(x) = f(xg) \ \forall \, g \in G(F), f \in \mathrm{Ind}_H^{G(F)} W, x \in G(F).$$

We may also write $(\mathrm{Ind}_H^{G(F)} \sigma, \mathrm{Ind}_H^{G(F)} W)$ instead of $(R, \mathrm{Ind}_H^{G(F)} W)$.

The *compact induction* of (σ, W) from H to $G(F)$ is the subrepresentation $(R, \text{c-ind}_H^{G(F)} W)$ of $(R, \mathrm{Ind}_H^{G(F)} W)$ consisting of functions $f \in \mathrm{Ind}_H^{G(F)} W$ whose support has compact image in $H\backslash G(F)$. We may also write $(\text{c-ind}_H^{G(F)} \sigma, \text{c-ind}_H^{G(F)} W)$ instead of $(R, \text{c-ind}_H^{G(F)} W)$.

For the smooth induction, we are particularly interested in the following special case.

Definition 1.2.5. Let $P \subset G$ be a parabolic subgroup of G with Levi decomposition $P = M \ltimes N$. Let (σ, W) be a smooth representation of the Levi subgroup $M(F)$. The (not normalized) *parabolic induction* $(\mathrm{Ind}_{P(F)}^{G(F)} \sigma, \mathrm{Ind}_{P(F)}^{G(F)} W)$ is defined by inflating (i.e. extending) the representation σ to a representation of $P(F)$ that is trivial on $N(F)$ and then inducing the resulting representation from $P(F)$ to $G(F)$.

Remark 1.2.6. We caution the reader that some authors normalize the parabolic induction by replacing $\sigma(m)$ by $\sigma(m) \left|\det \mathrm{Ad}_{\mathrm{Lie}(N)(F)}(m)\right|^{1/2}$ for $m \in M(F)$. This normalized induction preserves unitary. However, for our applications, both parabolic inductions, the normalized and the unnormalized one, work equally well.
[[todo: properly introduce normalized induction and state preservation of unitarity for Tasho's lecture series]]

This allows us to define supercuspidal representations.

Definition 1.2.7. An irreducible smooth representation (π, V) of $G(F)$ is called *supercuspidal* if for all proper parabolic subgroups $P \subsetneq G$ with Levi subgroup M and all irreducible smooth representations (σ, W) of $M(F)$ the representation (π, V) is not a subrepresentation of $(\mathrm{Ind}_{P(F)}^{G(F)} \sigma, \mathrm{Ind}_{P(F)}^{G(F)} W)$.

The following fact explains why we call the supercuspidal representations the building blocks.

Fact 1.2.8. *Let (π, V) be an irreducible smooth representation of G. Then there exists a parabolic subgroup $P \subseteq G$ with Levi subgroup M and a supercuspidal representation (σ, W) of $M(F)$ such that (π, V) is a subrepresentation of $(\mathrm{Ind}_{P(F)}^{G(F)} \sigma, \mathrm{Ind}_{P(F)}^{G(F)} W)$.*

It is a folklore conjecture that all supercuspidal representations arise via compact induction from a representation of a compact-mod-center open subgroup. In this survey we will outline how to construct all supercuspidal representations via compact induction under some mild tameness assumptions. In order to do this, we will need to introduce some additional structure theory. Though before doing so in the next section, let us mention the analogous definition of supercuspidal representations in the finite group case for later use.

Definition 1.2.9. Let H be the $\mathbb{F}_q$-points of a reductive group. An irreducible representation (π, V) of H is called *cuspidal* if the following equivalent conditions are satisfied:

(a) There does not exist a proper parabolic subgroup $P = MN$ of H and a representation (σ, W) of a Levi subgroup M such that V embeds into the induced representation $(\mathrm{Ind}_P^H \sigma, \mathrm{Ind}_P^H W)$.

(b) There does not exist a proper parabolic subgroup P of H with unipotent radical N such that the space of N-fixed vectors V^N is non-trivial.

We conclude this section by stating an equivalent definition of supercuspidal representations, for which we first introduce the contragredient representation.

Definition 1.2.10. Let (π, V) be a smooth representation of $G(F)$. We denote by V^* the dual vector space of V with the (often not smooth) representation π^* given by

$$\pi^*(g)(\lambda)(v) = \lambda(\pi(g^{-1})v) \quad \text{for } g \in G(F), \lambda \in V^*, v \in V.$$

The *contragredient representation* $(\widetilde{\pi}, \widetilde{V})$ is the restriction of the representation (π^*, V^*) to the subspace of smooth vectors $\widetilde{V} := \bigcup_K (V^*)^K$, where the union runs over all compact open subgroups K of $G(F)$.

Fact 1.2.11. *An irreducible smooth representations (π, V) of $G(F)$ is supercuspidal if and only if the image in $G(F)/Z(G(F))$ of the support of the function*

$$\begin{array}{rcl} G(F) & \to & \mathbb{C} \\ g & \mapsto & \lambda(\pi(g)v) \end{array}$$

is compact for all $v \in V, \lambda \in \widetilde{V}$, where $Z(G(F))$ denotes the center of $G(F)$. Equivalently, we may ask this condition to be satisfied only for some $0 \neq v \in V$ and $0 \neq \lambda \in \widetilde{V}$.

2 Moy–Prasad filtration and Bruhat–Tits theory

The Moy–Prasad filtration is a decreasing filtration of $G(F)$ by compact open subgroups that are normal inside each other and whose intersection is trivial. It is a refinement and generalization of the congruence filtration of $\mathrm{GL}_n(F)$. One usually starts with the definition of a Bruhat–Tits building that Bruhat and Tits ([BT72, BT84]) attached to the reductive group G over F in 1972/1984, and then to each point in the Bruhat–Tits building, Moy and Prasad ([MP94, MP96]) associated in 1994/1996 a filtration by compact open subgroups. In this survey, we will take a different approach and first introduce the Moy–Prasad filtration and use it to define the Bruhat–Tits building.

We first introduce some notation that we use throughout the remainder of the survey. For every finite extension E of F, we denote the ring of integers of E by $\mathcal{O}_E$ and a uniformizer by ϖ_E. We might drop the index E if $E = F$. We also fix a valuation $\mathrm{val} : F \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$.

2.1 The split case

We assume in this subsection that G is split over F. Let T be a split maximal torus of G and denote by $\Phi(G, T)$ the root system of G with respect to T. We recall that a *Chevalley system* $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$ consists of a non-trivial element X_α in the one dimensional F-vector root subspace $\mathfrak{g}_\alpha(F) \subset \mathfrak{g}(F)$ for each root α of G with respect to T such that

$$\mathrm{Ad}(w_\beta)(X_\alpha) = \pm X_{s_\beta(\alpha)} \,, \forall \alpha, \beta \in \Phi(G, T),$$

where w_β is an element of the normalizer $N_G(T)(F)$ of T in G determined by X_β whose image in the Weyl group $(N_G(T)/T)(F)$ is the simple reflection s_β corresponding to β. For example, if $G = \mathrm{SL}_2$ and $X_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $w_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In general w_β is defined as follows. For every root β, we let $x_\beta : \mathbb{G}_a \xrightarrow{\simeq} U_\beta$ be the isomorphism between the additive group $\mathbb{G}_a$ and the root group $U_\beta \subset G$ attached to β whose derivative sends $1 \in F = \mathbb{G}_a(F)$ to X_β. Then

$$w_\beta = x_\beta(1)x_{-\beta}(\epsilon)x_\beta(1)$$

where $\epsilon \in \{\pm 1\}$ is the unique element for which $x_\beta(1)x_{-\beta}(\epsilon)x_\beta(1)$ lies in the normalizer of T.

For example, for GL_n the collection $\{X_{\alpha_{i,j}}\}_{1\leq i,j\leq n; i\neq j}$ consisting of the matrices with all entries zero except for a one at position (i, j) forms a Chevalley system.

This allows us to make the following definition, but we warn the reader that we have not seen anyone else use the terminology "BT triple".

Notation 2.1.1. *A BT triple* (T, X_α, x_{BT}) *consists of*

(i) a split maximal torus T *of* G*,*

(ii) a Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$*, and*

(iii) $x_{BT} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} := \mathrm{Hom}_F(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Here $\mathbb{G}_m$ denotes the multiplicative group scheme and Hom_F denotes homomorphisms in the category of F-group schemes. Then $X_*(T) := \mathrm{Hom}_F(\mathbb{G}_m, T)$ is a free $\mathbb{Z}$-module, hence $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite-dimensional real vector space. Moreover, we have a bilinear pairing between $X^*(T) := \mathrm{Hom}_F(T, \mathbb{G}_m)$ and $X_*(T) = \mathrm{Hom}_F(\mathbb{G}_m, T)$ obtained by identifying $\mathrm{Hom}_F(\mathbb{G}_m, \mathbb{G}_m)$ with $\mathbb{Z}$. We extend this map $\mathbb{R}$-linearly in the second factor to obtain a map

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}.$$

In particular, we may pair x_{BT} with a root $\alpha \in \Phi(G, T)$ to obtain a real number $\langle \alpha, x_{BT} \rangle$.

We now fix a BT triple $x = (T, \{X_\alpha\}, x_{BT})$ and define the Moy–Prasad filtration attached to it.

Filtration of the torus.

We set

$$T(F)_0 = \{t \in T(F) \mid \mathrm{val}(\chi(t)) = 0 \ \forall \chi \in X^*(T) = \mathrm{Hom}_F(T, \mathbb{G}_m)\},$$

which is the maximal bounded subgroup of $T(F)$. For $r \in \mathbb{R}_{>0}$, we define

$$T(F)_r = \{t \in T(F)_0 \mid \mathrm{val}(\chi(t) - 1) \geq r \ \forall \chi \in X^*(T)\}.$$

For example, if $G = \mathrm{GL}_n$ and T is the torus consisting of diagonal matrices, then $T(F)_0$ consists of diagonal matrices whose entries are all in the invertible element $\mathcal{O}^\times$ of $\mathcal{O}$ and $T(F)_r$ consists of diagonal matrices whose entries are all in $1 + \varpi^{\lceil r \rceil}\mathcal{O}$, where we recall that ϖ is a uniformizer of F.

Filtration of the root groups.

Let $\alpha \in \Phi(G, T)$. We recall that the isomorphism $x_\alpha : \mathbb{G}_a \to U_\alpha$ is defined by requiring its derivative dx_α to send $1 \in F = \mathbb{G}_a(F)$ to X_α. For $r \in \mathbb{R}_{\geq 0}$, we define the filtration subgroups of $U_\alpha(F)$ as follows

$$U_\alpha(F)_{x,r} := x_\alpha(\varpi^{\lceil r - \langle \alpha, x_{BT} \rangle \rceil}\mathcal{O}).$$

Let us consider the example of $G = \mathrm{SL}_2$ and T the torus consisting of diagonal matrices.

Example 1. Let x_1 be the Bruhat–Tits triple $(T, \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}, 0)$. Let α correspond to the map $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^2$ for $t \in F^\times$. Then $x_\alpha(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ for $y \in F = \mathbb{G}_a(F)$ and

$$U_\alpha(F)_{x_1,r} = \begin{pmatrix} 1 & \varpi^{\lceil r \rceil}\mathcal{O} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_{-\alpha}(F)_{x_1,r} = \begin{pmatrix} 1 & 0 \\ \varpi^{\lceil r \rceil}\mathcal{O} & 1 \end{pmatrix}.$$

Example 2. Let x_2 be the Bruhat–Tits triple $(T, \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}, \frac{1}{4}\check{\alpha})$, where $\check{\alpha}$ is the coroot of α, i.e., the element of $X_*(T)$ that satisfies $\check{\alpha}(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ for $t \in F^\times = \mathbb{G}_m(F)$.

Then

$$U_\alpha(F)_{x_2,r} = \begin{pmatrix} 1 & \varpi^{\lceil r-\frac{1}{2}\rceil}\mathcal{O} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_{-\alpha}(F)_{x_2,r} = \begin{pmatrix} 1 & 0 \\ \varpi^{\lceil r+\frac{1}{2}\rceil}\mathcal{O} & 1 \end{pmatrix}.$$

Filtration of $G(F)$.

We define the filtration subgroup $G(F)_{x,r}$ of $G(F)$ for $r \in \mathbb{R}_{\geq 0}$ to be the subgroup generated by $T(F)_r$ and $U_\alpha(F)_{x,r}$ for all roots α, i.e.

$$G(F)_{x,r} = \langle T(F)_r, U_\alpha(F)_{x,r} \,|\, \alpha \in \Phi(G,T) \rangle \,.$$

If the ground field F is clear from the context, we may also abbreviate $G(F)_{x,r}$ by $G_{x,r}$.

In the example of $G = \mathrm{SL}_2$ for the two Bruhat–Tits triples above, we have for $r > 0$

$$G_{x_1,0} = \mathrm{SL}_2(\mathcal{O})\,, \;\; G_{x_1,r} = \begin{pmatrix} 1+\varpi^{\lceil r\rceil}\mathcal{O} & \varpi^{\lceil r\rceil}\mathcal{O} \\ \varpi^{\lceil r\rceil}\mathcal{O} & 1+\varpi^{\lceil r\rceil}\mathcal{O} \end{pmatrix}_{\det=1}$$

and

$$G_{x_2,0} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix}_{\det=1}, \;\; G_{x_2,r} = \begin{pmatrix} 1+\varpi^{\lceil r\rceil}\mathcal{O} & \varpi^{\lceil r-\frac{1}{2}\rceil}\mathcal{O} \\ \varpi^{\lceil r+\frac{1}{2}\rceil}\mathcal{O} & 1+\varpi^{\lceil r\rceil}\mathcal{O} \end{pmatrix}_{\det=1}.$$

Filtration of $\mathfrak{g}(F)$ and $\mathfrak{g}^*(F)$.

One can analogously define a filtration $\mathfrak{g}_{x,r}$ of the Lie algebra $\mathfrak{g}(F)$ and a filtration $\mathfrak{g}^*_{x,r}$ of the F-linear dual $\mathfrak{g}^*(F)$ of the Lie algebra $\mathfrak{g}(F)$ as follows. Let r be a real number, and recall that we write $\mathfrak{t}$ for the Lie algebra of the torus T. Then we set

$$\mathfrak{t}(F)_r = \{X \in \mathfrak{t}(F) \,|\, \mathrm{val}(d\chi(X)) \geq r \;\forall\, \chi \in X^*(T)\},$$

where $d\chi$ denotes the derivative of χ,

$$\mathfrak{g}_\alpha(F)_{x,r} = \varpi^{\lceil r-\langle\alpha,x_{BT}\rangle\rceil}\mathcal{O}\,X_\alpha \subset \mathfrak{g}_\alpha(F)$$

for $\alpha \in \Phi(G,T)$, and

$$\mathfrak{g}(F)_{x,r} = \mathfrak{t}(F)_r \oplus \bigoplus_{\alpha\in\Phi(G,T)} \mathfrak{g}_\alpha(F)_{x,r}.$$

We define the filtration subspace $\mathfrak{g}^*(F)_{x,r}$ of the dual of the Lie algebra by

$$\mathfrak{g}^*(F)_{x,r} = \{X \in \mathfrak{g}^*(F) \,|\, X(Y) \in \varpi\mathcal{O} \;\text{ for all }\; Y \in \mathfrak{g}(F)_{x,s} \;\text{ with }\; s > -r\}.$$

If the ground field F is clear from the context, we may also abbreviate $\mathfrak{g}(F)_{x,r}$ and $\mathfrak{g}^*(F)_{x,r}$ by $\mathfrak{g}_{x,r}$ and $\mathfrak{g}^*_{x,r}$, respectively.

2.1.1 Properties of the Moy–Prasad filtration

Definition 2.1.2. A *parahoric* subgroup of G is a subgroup of the form $G_{x,0}$ for some BT triple x.

For $r \in \mathbb{R}_{\geq 0}$, we write $G_{x,r+} = \bigcup_{s>r} G_{x,s}$ and $\mathfrak{g}_{x,r+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$.

We collect a few facts about this filtration that demonstrate the richness of its structure.

Fact 2.1.3. *Let x be a BT triple.*

(a) $G_{x,r}$ is a normal subgroup of $G_{x,0}$ for all $r \in \mathbb{R}_{\geq 0}$.

(b) The quotient $G_{x,0}/G_{x,0+}$ is the group of the $\mathbb{F}_q$-points of a reductive group $\mathbb{G}_x$ defined over the residue field $\mathbb{F}_q$ of F.

(c) For $r \in \mathbb{R}_{>0}$, the quotient $G_{x,r}/G_{x,r+}$ is abelian and can be identified with an $\mathbb{F}_q$-vector space.

(d) Let $r > 0$. Since $G_{x,r}$ is a normal subgroup of $G_{x,0}$, the group $G_{x,0}$ acts on $G_{x,r}$ via conjugation. This action descends to an action of the quotient $G_{x,0}/G_{x,0+}$ on the vector space $G_{x,r}/G_{x,r+}$ and the resulting action is (the $\mathbb{F}_q$-points of) a linear algebraic action, i.e., corresponds to a morphism from $\mathbb{G}_x$ to $\mathrm{GL}_{\dim_{\mathbb{F}_q}(G_{x,r}/G_{x,r+})}$ over $\mathbb{F}_q$.

(e) We have the following isomorphism that is often referred to as the "Moy–Prasad isomorphism": $G_{x,r}/G_{x,r+} \simeq \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ for $r \in \mathbb{R}_{>0}$ and more general $G_{x,r}/G_{x,s} \simeq \mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}$ for $r, s \in \mathbb{R}_{>0}$ with $2r \geq s > r$.

In fact we have a rather good understanding of the representations occurring in (d). In [Fin21b] they are described explicitly in terms of Weyl modules. Previously they were also realized using Vinberg–Levy theory by Reeder and Yu ([RY14]), which was generalized in [Fin21b].

2.1.2 The Bruhat–Tits building

Definition 2.1.4. The *(reduced) Bruhat–Tits building* $\mathscr{B}(G, F)$ of G over F is as a set the quotient of the set of BT triples by the following equivalence relation: Two BT triple x_1 and x_2 are equivalent if and only if $G_{x_1,r} = G_{x_2,r}$ for all $r \in \mathbb{R}_{\geq 0}$.

As a consequence of the definition, for $x \in \mathscr{B}(G, T)$, we may write $G_{x,r}$ for the Moy–Prasad filtration attached to any BT triple in the equivalence class of x.

The Bruhat–Tits building $\mathscr{B}(G, F)$ admits an action of $G(F)$ that is determined by the property

$$G_{g.x,r} = gG_{x,r}g^{-1} \; \forall \, r \in \mathbb{R}_{\geq 0}, g \in G(F).$$

We will now equip the Bruhat–Tits building with more structure.

Apartments as affine spaces.

Definition 2.1.5. For a split maximal torus T, we call the subset of $\mathscr{B}(G,F)$ that can be represented by BT triples whose first entry is the given torus T, i.e.

$$\mathscr{A}(T,F) := \{(T, \{X_\alpha\}, x_{BT})\}/_\sim \subset \mathscr{B}(G,F)$$

the *apartment* of T.

We fix a split maximal torus T and a Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$. Then it turns out that every element in $\mathscr{A}(T,F)$ can be represented by a BT triple whose first two entries are the torus T and the fixed Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$. Moreover, two BT triples $(T, \{X_\alpha\}, x_{BT,1})$ and $(T, \{X_\alpha\}, x_{BT,2})$ are equivalent if and only if $x_{BT,2} - x_{BT,1}$ lies in the subspace $X_*(Z(G)) \otimes \mathbb{R}$, where $Z(G)$ denotes the center of G. Note that $X_*(Z(G)) \otimes \mathbb{R}$ is trivial when the center $Z(G)$ of G is finite. Thus the set $\mathscr{A}(T,F)$ is isomorphic to $X_*(T)\otimes\mathbb{R}/X_*(Z(G))\otimes\mathbb{R}$, and we use this isomorphism to equip $\mathscr{A}(T,F)$ with the structure of an affine space over the real vector space $X_*(T)\otimes\mathbb{R}/X_*(Z(G))\otimes\mathbb{R}$. While the isomorphism of $\mathscr{A}(T,F)$ with $X_*(T)\otimes\mathbb{R}/X_*(Z(G))\otimes\mathbb{R}$ depends on the choice of the Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$, the structure of $\mathscr{A}(T,F)$ as an affine space does not. In fact, the choice of a Chevalley system turns the affine space into a vector space by choosing a base point.

Polysimplicial structure on apartments.

Let T be a split maximal torus of G. For $\alpha \in \Phi(G,T)$, we define the following set of hyperplanes of the apartment $\mathscr{A}(T,F)$:

$$\Psi_\alpha := \left\{ \text{hyperplanes } H \subset \mathscr{A}(T,F) \text{ satisfying } \begin{array}{ll} U_\alpha(F)_{x,0} = U_\alpha(F)_{y,0} & \forall x,y \in H \\ U_\alpha(F)_{x,0} \neq U_\alpha(F)_{x,0+} & \forall x \in H \end{array} \right\}.$$

We set

$$\Psi := \bigcup_{\alpha\in\Phi(G,T)} \Psi_\alpha$$

and use these hyperplanes to turn the apartment $\mathscr{A}(T,F)$ into the geometric realization of a polysimplicial complex. This means the connected components of the complement of the union of the hyperplanes in Ψ are the maximal dimensional polysimplices, which are also called *chambers*.

Polysimplicial structure on the Bruhat–Tits building. The polysimplicial structure on the apartments yields a polysimplicial structure on the Bruhat–Tits building $\mathscr{B}(G,F)$, which satisfies the properties of an abstract building. In order to recall the notion of an abstract building, we need to introduce some notation following [KP23, §1.5].

A *chamber complex* is a polysimplicial complex $\mathscr{B}$ in which every facet is contained in a maximal facet, called *chamber*, and given two chambers C and C' there exists a sequence $C = C_1 \neq C_2 \neq C_3 \neq \ldots \neq C_n = C'$ such that $C_i \in \mathscr{B}, C_i \cap C_{i+1} \in \mathscr{B}$ and $\nexists C''$ with $C_i \cap C_{i+1} \subsetneq C'' \subsetneq C_i$ or $C_i \cap C_{i+1} \subsetneq C'' \subsetneq C_{i+1}$. A chamber complex is called *thick* if each facet of codimension one is the face of at least three chambers and is called *thin* if each facet of codimension one is the face of exactly two chambers.

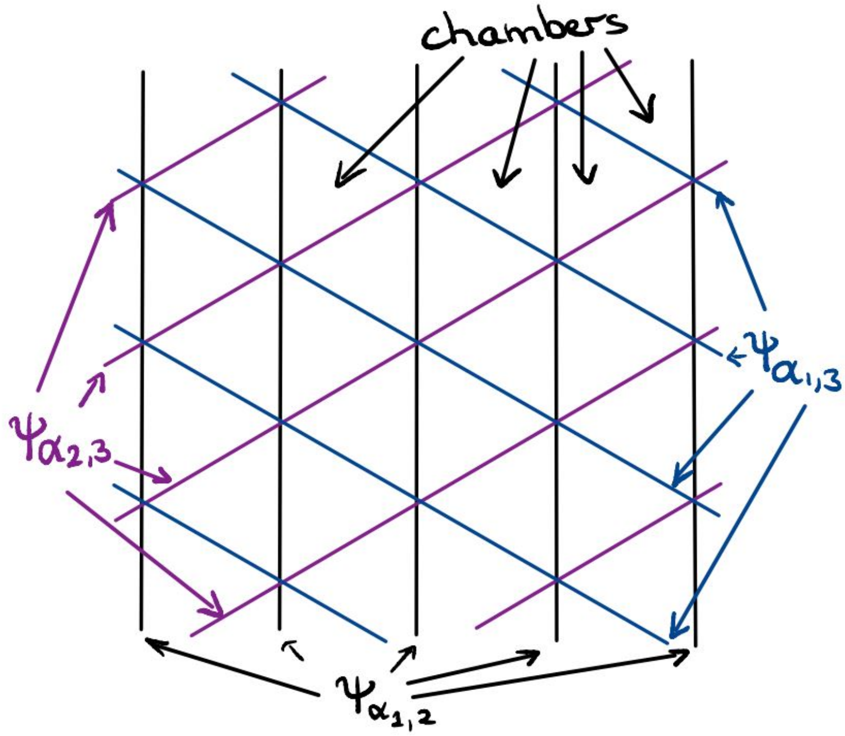

Figure 1: Excerpt of an apartment for SL_3 with hyperplanes, where $\alpha_{i,j}$ is the root corresponding to $\mathrm{diag}(t_1, t_2, t_3) \mapsto t_i t_j^{-1}$

Definition 2.1.6 (see [KP23, Definition 1.5.5])**.** A *building* is a chamber complex $\mathscr{B}$ equipped with a collection of subcomplexes, called *apartments*, satisfying the following axioms.

(i) $\mathscr{B}$ is a thick chamber complex.

(ii) Each apartment is a thin chamber complex.

(iii) Any two chambers belong to an apartment.

(iv) Given two apartments $\mathscr{A}_1$ and $\mathscr{A}_2$, and two facets $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{A}_1 \cap \mathscr{A}_2$, there exits an isomorphism $\mathscr{A}_1 \to \mathscr{A}_2$ of chamber complexes that leaves invariant $\mathscr{F}_1, \mathscr{F}_2$ and all of their faces.

Fact 2.1.7. *The Bruhat–Tits building $\mathscr{B}(G, F)$ with its apartments $\mathscr{A}(T, F)$ attached to maximal split tori T of G is a geometric realization of a building.*

2.2 The non-split (tame) case

We first assume that G splits over an unramified Galois field extension E over F. In that case all the above definitions can be descended to G by taking $\mathrm{Gal}(E/F)$-fixed points of the corresponding objects for G_E. More precisely, we set

$$G(F)_{x,r} = G(E)_{x,r}^{\mathrm{Gal}(E/F)},$$

where $G(E)_{x,r}$ is defined using the valuation on E that extends the valuation val on F. As in the split case, we may abbreviate $G(F)_{x,r}$ by $G_{x,r}$.

Via the action of $\mathrm{Gal}(E/F)$ on $G(E)$ and hence on its filtration subgroups, we obtain an action of $\mathrm{Gal}(E/F)$ on the Bruhat–Tits building $\mathscr{B}(G,E)$ and we define

$$\mathscr{B}(G,F) = \mathscr{B}(G,E)^{\mathrm{Gal}(E/F)}.$$

More generally, if we only assume that G splits over a tamely ramified Galois field extension E over F, then we have for $r > 0$

$$G_{x,r} = G(F)_{x,r} = G(E)_{x,r}^{\mathrm{Gal}(E/F)},$$

where $G(E)_{x,r}$ is defined using the valuation on E that extends the valuation val on F and $U_\alpha(E)_{x,r} = x_\alpha(\varpi_E^{\lceil e(r-\langle\alpha,x_{BT}\rangle)\rceil}\mathcal{O}_E)$ with e the ramification index of the field extension E over F. Defining the parahoric subgroup $G(F)_{x,0}$ is slightly more subtle in general. It is a finite index subgroup of $G(E)_{x,0}^{\mathrm{Gal}(E/F)}$. The parahoric subgroup $G(F)_{x,0}$ being occasionally a slightly smaller group than $G(E)_{x,0}^{\mathrm{Gal}(E/F)}$ will ensure that $G(F)_{x,0}/G(F)_{x,0+}$ are the $\mathbb{F}_q$-points of a connected reductive group rather than a potentially disconnected group. More precisely, the parahoric subgroup $G(F)_{x,0}$ is defined by

$$G_{x,0} = G(F)_{x,0} = G(E)_{x,0}^{\mathrm{Gal}(E/F)} \cap G(F)^0$$

for some explicitly constructed normal subgroup $G(F)^0 \subset G(F)$. We refer the interested reader to the literature, e.g., [KP23], for the precise definition of $G(F)^0$ and only note that $G(F)^0 = G(F)$ if G is simply connected semi-simple, e.g., for $G = \mathrm{SL}_n$ we have $\mathrm{SL}_n(F)^0 = \mathrm{SL}_n(F)$.

As in the unramified setting, using the action of $\mathrm{Gal}(E/F)$ on $G(E)$ and hence on its filtration subgroups, we obtain an action of $\mathrm{Gal}(E/F)$ on the Bruhat–Tits building $\mathscr{B}(G,E)$ and we define

$$\mathscr{B}(G,F) = \mathscr{B}(G,E)^{\mathrm{Gal}(E/F)}.$$

Similarly, we have for the Lie algebra

$$\mathfrak{g}_{x,r} = \mathfrak{g}(F)_{x,r} = (\mathfrak{g}(E)_{x,r})^{\mathrm{Gal}(E/F)}.$$

We note that the above definitions rely on the extension E over F being tame, but are independent of the choice of E.

Aside 2.2.1. If G splits only over a wildly ramified extension E/F, then the space of fixed vectors of the Galois action on the Bruhat–Tits building over E might be larger than the Bruhat–Tits building defined over F. We will not introduce the Bruhat–Tits building in that generality in this survey since we mostly restrict to the tame case, and we instead refer the interested reader to the literature, e.g., the original articles by Bruhat and Tits ([BT72, BT84]) and the recent book on this topic by Kaletha and Prasad [KP23].

The Moy–Prasad filtration still satisfies the nice properties as in Fact 2.1.3, i.e., more precisely

Fact 2.2.2. *Let $x \in \mathscr{B}(G, F)$.*

(a) $G_{x,r}$ is a normal subgroup of $G_{x,0}$ for all $r \in \mathbb{R}_{\geq 0}$.

(b) The quotient $G_{x,0}/G_{x,0+}$ is the group of the $\mathbb{F}_q$-points of a reductive group $\mathbb{G}_x$ defined over the residue field $\mathbb{F}_q$ of F.

(c) For $r \in \mathbb{R}_{>0}$, the quotient $G_{x,r}/G_{x,r+}$ is abelian and can be identified with an $\mathbb{F}_q$-vector space.

(d) Let $r > 0$. Since $G_{x,r}$ is a normal subgroup of $G_{x,0}$, the group $G_{x,0}$ acts on $G_{x,r}$ via conjugation. This action descends to an action of the quotient $G_{x,0}/G_{x,0+}$ on the vector space $G_{x,r}/G_{x,r+}$ and the resulting action is (the $\mathbb{F}_q$-points of) a linear algebraic action, i.e., corresponds to a morphism from $\mathbb{G}_x$ to $\mathrm{GL}_{\dim_{\mathbb{F}_q}(G_{x,r}/G_{x,r+})}$ over $\mathbb{F}_q$.

(e) Under the assumption that G splits over a tamely ramified field extension of F, we have the following "Moy–Prasad isomorphism": $G_{x,r}/G_{x,r+} \simeq \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ for $r \in \mathbb{R}_{>0}$ and more general $G_{x,r}/G_{x,s} \simeq \mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}$ for $r, s \in \mathbb{R}_{>0}$ with $2r \geq s > r$.

Definition 2.2.3. For a maximal split torus S of G, we choose a maximal torus $T \subset G$ containing S and call the subset $\mathscr{A}(S, F) := \mathscr{A}(T, E)^{\mathrm{Gal}(E/F)}$ of $\mathscr{B}(G, F)$, sometimes also denoted by $\mathscr{A}(T, F)$, the *apartment* of S (or of T).

Note that $\mathscr{A}(S, F) = \mathscr{A}(T, E)^{\mathrm{Gal}(E/F)}$ is independent on the choice of maximal torus T containing S, i.e., the apartment of S is well defined. The apartment $\mathscr{A}(S, F)$ is an affine space over the real vector space $X_*(S) \otimes \mathbb{R}/X_*(Z(G)) \otimes \mathbb{R}$. We will equip the apartment with a polysimplicial structure analogous to the split case.

To do this, let S be a maximal split torus of G and T a maximal torus of G containing S that splits over a tame extension E of F. We denote by $\Phi(G, S)$ the relative root system, i.e., the restrictions of the (absolute) roots $\Phi(G, T) \subset X^*(T_E)$ to S that are non-trivial, or in other words the non-trivial weights of the action of S on the Lie algebra of G. For $a \in \Phi(G, S)$, we have the root group U_a, which is the unique smooth closed subgroup of G that is normalized by S and on whose Lie algebra S acts by positive integer multiples of a.

We have $U_a(F) = (\prod_\alpha U_\alpha(E))^{\mathrm{Gal}(E/F)}$, where α runs over the roots of $\Phi(G,T)$ that restrict to a positive integer multiple of a. We set

$$U_{a,x,r} := \left(\prod_\alpha U_\alpha(E)_{x,r}\right)^{\mathrm{Gal}(E/F)}$$

for $r \in \mathbb{R}_{\geq 0}$ using the same normalization as for the definition of $G_{x,r}$ above.

Now we can define a set of hyperplanes of the apartment $\mathscr{A}(S,F)$ for $a \in \Phi(G,S)$:

$$\Psi_a := \left\{ \text{hyperplanes } H \subset \mathscr{A}(S,F) \text{ satisfying } \begin{array}{ll} U_{a,x,0} = U_{a,y,0} & \forall x,y \in H \\ U_{a,x,0} \neq U_{a,x,0+} & \forall x \in H \end{array} \right\}.$$

We set

$$\Psi := \bigcup_{a \in \Phi(G,S)} \Psi_a$$

and use these hyperplanes to turn the apartment $\mathscr{A}(S,F)$ into the geometric realization of a polysimplicial complex. This polysimplicial structure on the apartments yields a polysimplicial structure on the Bruhat–Tits building $\mathscr{B}(G,F)$ and as in the split case, we have the following result.

Fact 2.2.4. *The Bruhat–Tits building $\mathscr{B}(G,F)$ with its apartments $\mathscr{A}(S,F)$ attached to maximal split tori S of G is a geometric realization of a building.*

We record the following fact that will become useful later when constructing supercuspidal representations.

Fact 2.2.5. *Let $x,y \in \mathscr{A}(S,F) \subset \mathscr{B}(G,F)$. Then the image of $G_{x,0} \cap G_{y,0}$ in $G_{y,0}/G_{y,0+}$ is a parabolic subgroup $P_{x,y}$ and the image of $G_{x,0+} \cap G_{y,0}$ in $G_{y,0}/G_{y,0+}$ is the unipotent radical of $P_{x,y}$. If $x \neq y$ and y is a vertex, i.e., a polysimplex of minimal dimension, then $P_{x,y}$ is a proper parabolic subgroup.*

2.3 The enlarged Bruhat–Tits building

In some circumstances it is more convenient to work with the enlarged Bruhat–Tits building. The *enlarged Bruhat–Tits building* $\widetilde{\mathscr{B}}(G,F)$ is defined as the product of the reduced Bruhat–Tits building $\mathscr{B}(G,F)$ with $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e.,

$$\widetilde{\mathscr{B}}(G,F) = \mathscr{B}(G,F) \times X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}.$$

This means that if the center of G is finite, then the reduced and the non-reduced Bruhat–Tits buildings are the same. In general, an important difference is that stabilizers in $G(F)$ of points in the enlarged Bruhat–Tits building are compact while stabilizers of points in the reduced Bruhat–Tits building contain the center of $G(F)$ and are compact-mod-center. For the enlarged building, the apartments $\widetilde{\mathscr{A}}(S,F)$ correspond to maximal split tori S and are affine spaces under the action of $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. For a point $x \in \widetilde{\mathscr{B}}(G,F)$ we denote by $[x]$ the image of x in $\mathscr{B}(G,F)$ (by projection to the first factor) and we define $G_{x,r} := G_{[x],r}$ for $r \in \mathbb{R}_{\geq 0}$ and $\mathfrak{g}_{x,r} := \mathfrak{g}_{[x],r}$ and $\mathfrak{g}^*_{x,r} := \mathfrak{g}^*_{[x],r}$ for $r \in \mathbb{R}$.

2.4 The depth of a representation

The Moy–Prasad filtration allows us to introduce the notion of the depth of an irreducible smooth representation, initially defined by Moy and Prasad in [MP94, MP96]. Our definition is slightly different but equivalent to theirs.

Definition 2.4.1. Let (π, V) be an irreducible smooth representation of G. The *depth* of (π, V) is the smallest non-negative real number r such that $V^{G_{x,r+}} \neq \{0\}$ for some $x \in \mathscr{B}(G, F)$.

3 Construction of supercuspidal representations

3.1 A non-exhaustive overview of some historic developments

In 1977, a Symposium in Pure Mathematics was held in Corvallis that led to famous Proceedings. One of the articles in the Proceedings was entitled "Representations of p-adic groups: A survey", written by Cartier ([Car79b]). We quote from the introduction of this article:

> "The main goal of this article will be the description and study of the principal series and the spherical functions. There shall be almost no mention of two important lines of research which are still actively pursued today:
> (a) [...]
> (b) *Explicit construction of absolutely cuspidal representations [nowadays usually called "supercuspidal representations"].* Here important progress has been made by Shintani [Shi68], Gérardin [Gér75] and Howe (forthcoming papers in the Pacific J. Math.). One can expect to meet here difficult and deep arithmetical questions which are barely uncovered."

Since then, mathematicians have tried to construct the mysterious supercuspidal representations. To mention a few, in 1979, Carayol ([Car79a]) gave a construction of all supercuspidal representations of the general linear group $\mathrm{GL}_n(F)$ for n a prime number different from p, the residue field characteristic of F. In 1986, Moy ([Moy86a]) proved that Howe's construction ([How77]) from the 1970s exhausts all supercuspidal representations of $\mathrm{GL}_n(F)$ if n is coprime to p. In the early 1990s, Bushnell and Kutzko extended these constructions to obtain all supercuspidal representations of $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$ for arbitrary n ([BK93, BK94]). Similar methods have been exploited by Stevens ([Ste08]) around 15 years ago to construct all supercuspidal representations of classical groups for $p \neq 2$, i.e., orthogonal, symplectic and unitary groups. His work was preceded by a series of partial results by Moy ([Moy86b] for $\mathrm{U}(2, 1)$, [Moy88] for GSp_4), Morris ([Mor91] and [Mor92]) and Kim ([Kim99]). Moreover, Zink ([Zin92]) treated division algebras over non-archimedean local fields of characteristic zero, Broussous ([Bro98]) treated division algebras without restriction on the characteristic, and Sécherre and Stevens ([SS08]) completed the case of all inner forms of $\mathrm{GL}_n(F)$ about 15 years ago.

In order to achieve progress for arbitrary reductive groups, the work of Moy and Prasad based on the work of Bruhat and Tits, introduced in Section 2, was pivotal. The Moy–Prasad filtration allowed Moy and Prasad introduced in [MP94, MP96] to introduce the notion of *depth* of a representation, see Definition 2.4.1, and gave a classification of depth-zero representations. Moy and Prasad showed, roughly speaking, that depth-zero representations correspond to representations of finite groups of Lie type. A similar result was obtained shortly afterwards using different techniques by Morris ([Mor99]). We will discuss depth-zero representations in more detail in Section 3.3.

In 1998, Adler ([Adl98]) used the Moy–Prasad filtration to suggest a construction of positive-depth supercuspidal representations for general p-adic groups that split over a tamely ramified extension of F, which was generalized by Yu ([Yu01]) in 2001. Since then, Yu's construction has been widely used in the representation theory of p-adic groups as well as for applications thereof. We will sketch Yu's construction in Section 3.8.

Kim ([Kim07]) achieved the subsequent breakthrough in 2007 by proving that if F has characteristic zero and the prime number p is "very large", then all supercuspidal representations arise from Yu's construction. Recently, in 2021, Fintzen ([Fin21d]) has shown via very different techniques that Yu's construction provides us with all supercuspidal representations only under the minor assumption that p does not divide the order of the absolute Weyl group of the (tame) p-adic group. In particular, the result also holds for fields F of positive characteristic. Based on [Fin21c], we expect this result to be essentially optimal (when considering

type	$A_n\ (n \geq 1)$	$B_n, C_n\ (n \geq 2)$	$D_n\ (n \geq 3)$	E_6	E_7
order	$(n+1)!$	$2^n \cdot n!$	$2^{n-1} \cdot n!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$

type	E_8	F_4	G_2
order	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2$	$2^2 \cdot 3$

Table 1: Order of irreducible Weyl groups ([Bou02, VI.4.5-VI.4.13])

also types for non-supercuspidal Bernstein blocks and treating all inner forms together), and it is exciting research in progress to construct the remaining supercuspidal representations for small primes. For this survey, we will focus on the known construction of supercuspidal representations under the assumption that p does not divide the order of the absolute Weyl group.

However, it was recently suggested by Fintzen, Kaletha and Spice ([FKS23]) to twist Yu's construction by a quadratic character, i.e., a character of an appropriate compact open subgroup appearing in the construction of supercuspidal representations that takes values in $\{\pm 1\}$, see Section 3.10. While on first glance this just looks like changing the parametrization of supercuspidal representations, the existence of the quadratic character has far-reaching consequences. For example, it allowed to calculate formulas for the Harish-Chandra character of these supercuspidal representations ([FKS23, Spi]), to specify a candidate for the local Langlands correspondence for non-singular supercuspidal representations ([Kal]) and to prove that the local Langlands correspondence for regular supercuspidal representations introduced by Kaletha ([Kal19]) satisfies the desired character identities ([FKS23]). It is also crucial for obtaining isomorphisms between Hecke algebras attached to Bernstein blocks of arbitrary depth and those of depth-zero ([AFMOa, AFMOb]), the topic discussed below in Section 4.7.

3.2 Generalities about the construction of supercuspidal representations

All representations are always taken to be smooth and to have complex coefficients unless stated otherwise. Possible references for the facts discussed below include [BH06, DeB16, Ren10, Vig96].

It is a folklore conjecture that all supercuspidal irreducible representations arise via compact induction from a representation of a compact-mod-center open subgroup of $G(F)$, and all constructions mentioned above proceed in this way. It is a nice exercise to use Fact 1.2.11 to deduce the following lemma.

Lemma 3.2.1. *Let K be a compact-mod-center open subgroup of $G(F)$ and let ρ be an irreducible representation of K. If the compact induction $\text{c-ind}_K^{G(F)} \rho$ of ρ from K to $G(F)$ is irreducible, then $\text{c-ind}_K^{G(F)} \rho$ is a supercuspidal representation of $G(F)$.*

Thus in order to construct supercuspidal representations, it suffices to construct pairs (K, ρ) of compact-mod-center open subgroups and irreducible representations thereof such that $\text{c-ind}_K^{G(F)} \rho$ is irreducible. The standard approach to show the latter is via Lemma 3.2.3 below, which we will demonstrate in examples below. In order to state the fact, we need to introduce some notation.

Let K be a compact-mod-center open subgroup of $G(F)$ that contains the center $Z(G(F))$ of $G(F)$ and let (ρ, W) be a smooth representation of K.

Notation 3.2.2. *For $g \in G(F)$, we write ${}^g\rho$ for the representation of ${}^gK := gKg^{-1}$ satisfying ${}^g\rho(h) = \rho(g^{-1}hg)$ for $h \in {}^gK$.*

We say that g intertwines (ρ, W) if $\text{Hom}_{{}^gK\cap K}({}^g\rho|_{{}^gK\cap K}, \rho|_{{}^gK\cap K}) \neq \{0\}$.

Lemma 3.2.3. *Let K be an open subgroup of $G(F)$ that contains and is compact modulo the center $Z(G(F))$ of $G(F)$. Let (ρ, W) be an irreducible representation of K. Suppose $g \in G(F)$ intertwines (ρ, W) if and only if $g \in K$. Then $\text{c-ind}_K^{G(F)} \rho$ is irreducible.*

In order to prove this lemma, let us note a helpful result, the Mackey decomposition, whose proof is a nice exercise using the definition of the compact induction.

Lemma 3.2.4 (Mackey decomposition)**.** *If K' is a compact-mod-center open subgroup of $G(F)$, then the restriction of $\text{c-ind}_K^{G(F)} \rho$ to K' decomposes as a representation of K' as follows*

$$(\text{c-ind}_K^{G(F)} \rho)|_{K'} = \bigoplus_{g \in K' \backslash G(F)/K} \text{c-ind}_{{}^gK\cap K'}^{K'} {}^g\rho|_{{}^gK\cap K'} .$$

Proof. Left to the reader. □

Proof of Lemma 3.2.3. First note that (ρ, W) is a K-subrepresentation of $\left(\left(\text{c-ind}_K^{G(F)} \rho\right)|_K, \text{c-ind}_K^{G(F)} W\right)$ via the embedding

$$w \quad \mapsto \quad f_w : g \mapsto \begin{cases} \rho(g)w & g \in K \\ 0 & g \notin K \end{cases},$$

and the image of W in $\text{c-ind}_K^{G(F)} W$ under this embedding generates the latter as a $G(F)$-representation. We claim that this is up to scalar the only embedding of ρ into $\left(\text{c-ind}_K^{G(F)} \rho\right)|_K$. This follows from:

$$\begin{aligned} \text{Hom}_K\left(\rho, (\text{c-ind}_K^{G(F)} \rho)|_K\right) &\simeq \bigoplus_{g \in K\backslash G(F)/K} \text{Hom}_K\left(\rho, \text{c-ind}^K_{{}^gK\cap K} {}^g\rho|_{{}^gK\cap K}\right) \\ &\simeq \bigoplus_{g \in K\backslash G(F)/K} \text{Hom}_{{}^gK\cap K}\left(\rho|_{{}^gK\cap K}, {}^g\rho|_{{}^gK\cap K}\right) \\ &= \text{Hom}_K\left(\rho|_{K\cap K}, \rho|_{K\cap K}\right) \simeq \mathbb{C}, \end{aligned}$$

where the first isomorphism results from the Mackey decomposition (Lemma 3.2.4) and the second from Frobenius reciprocity as the involved compact induction agrees with the smooth induction.

Suppose now that $V \subset \text{c-ind}_K^{G(F)} W$ is a non-trivial $G(F)$-subrepresentation. This implies

$$\{0\} \not\simeq \text{Hom}_G\left(V, \text{c-ind}_K^{G(F)} W\right) \subset \text{Hom}_G\left(V, \text{Ind}_K^{G(F)} W\right) \simeq \text{Hom}_K(V, W),$$

where Ind denotes the smooth induction. Note that $Z(G(F))$ acts via the central character of ρ on $\text{c-ind}_K^{G(F)} W$ and hence on V. Thus, as a K-representation, V is a direct sum of irreducible K-representations. Therefore the above observation $\text{Hom}_K(V, W) \not\simeq \{0\}$ implies that W is isomorphic to a subrepresentation of V. By the uniqueness up to scalar of the embedding of W into $\text{c-ind}_K^{G(F)} W$ as K-representations, we deduce that V contains the above image of W, which generates $\text{c-ind}_K^{G(F)} W$ as a $G(F)$-representation. Since V is a $G(F)$-representation, we obtain $V = \text{c-ind}_K^{G(F)} W$. □

3.3 Depth-zero supercuspidal representations

In this section, we consider the special case of depth-zero supercuspidal representations. The following theorem is due to Moy and Prasad ([MP94, MP96]) and a different proof was later given by Morris ([Mor99]).

Theorem 3.3.1 ([MP94, MP96, Mor99])**.** *Let $x \in \mathscr{B}(G, F)$ be a vertex. Let (ρ, V_ρ) be an irreducible smooth representation of the stabilizer G_x of x that is trivial on $G_{x,0+}$ and such that $\rho|_{G_{x,0}}$ is a cuspidal representation of the reductive group $G_{x,0}/G_{x,0+}$. Then $\text{c-ind}_{G_x}^{G(F)} \rho$ is a supercuspidal irreducible representation of $G(F)$.*

The above authors also showed that all depth-zero supercuspidal (irreducible smooth) representations are of the form as in Theorem 3.3.1.

Remark 3.3.2. Requiring that $\rho|_{G_{x,0}}$ is a cuspidal representation in Theorem 3.3.1 is equivalent to asking that $\rho|_{G_{x,0}}$ contains an irreducible, cuspidal representation of $G_{x,0}/G_{x,0+}$, because $\rho|_{G_{x,0}}$ is a direct sum of irreducible representations that are transitively permuted by the action of G_x.

Proof of Theorem 3.3.1.
By Lemmata 3.2.1 and 3.2.3 it suffices to show that an element $g \in G(F)$ intertwines (ρ, V_ρ) if and only if $g \in G_x$. Since all $g \in G_x$ intertwine (ρ, V_ρ), it remains to show the other direction of the implication. Hence we assume $g \in G(F)$ intertwines (ρ, V_ρ), i.e., we can choose a nontrivial element

$$f \in \mathrm{Hom}_{G_x \cap gG_xg^{-1}}({}^g\sigma, \sigma) \not\supsetneq \{0\}.$$

Since σ is trivial when restricted to $G_{x,0+}$, the representation ${}^g\sigma$ is trivial when restricted to $gG_{x,0+}g^{-1} = G_{g.x,0+}$. Hence $G_{g.x,0+} \cap G_{x,0}$ acts trivially on the image $\mathrm{Im}(f)$ of f. If $g \notin G_x$, then $g.x \neq x$ and hence by Fact 2.2.5, the image of $G_{g.x,0+} \cap G_{x,0}$ in $G_{x,0}/G_{x,0+}$ is the unipotent radical N of a proper parabolic subgroup of $G_{x,0}/G_{x,0+}$. Thus

$$\{0\} \not\supsetneq \mathrm{Im}(f) \subset V_\rho^N,$$

which contradicts that (ρ, V_ρ) is cuspidal. □

3.4 An example of a positive-depth supercuspidal representations

From now on we fix an additive character $\varphi : F \to \mathbb{C}^\times$ (i.e., a group homomorphism from the group F (equipped with addition) to the group $\mathbb{C}^\times$ (equipped with multiplication)) that is nontrivial on $\mathcal{O}$ and trivial on $\varpi\mathcal{O}$.

We start with an example of a positive-depth supercuspidal representation that we will stepwise generalize. Let $G = \mathrm{SL}_2$. Consider the point $x_2 \in \mathscr{B}(\mathrm{SL}_2, F)$ introduced in Example 2 on page 7, which is the unique point x_2 for which

$$G_{x_2,r} = \begin{pmatrix} 1+\varpi^{\lceil r\rceil}\mathcal{O} & \varpi^{\lceil r-\frac{1}{2}\rceil}\mathcal{O} \\ \varpi^{\lceil r+\frac{1}{2}\rceil}\mathcal{O} & 1+\varpi^{\lceil r\rceil}\mathcal{O} \end{pmatrix}_{\det=1} \quad \text{for } r \in \mathbb{R}_{>0}.$$

Let $K = \{\pm \mathrm{Id}\}G_{x_2,\frac{1}{2}}$. We define the representation $(\rho, \mathbb{C})$, i.e., the morphism $\rho : K \to \mathbb{C}^\times$ by requiring

$$\rho(\pm \mathrm{Id}) = 1 \quad \text{and} \quad \rho\left(\begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{pmatrix}\right) = \varphi(b+c)$$

for all $a, b, c, d \in \mathcal{O}$ with $(1+\varpi a)(1+\varpi d) - \varpi bc = 1$. Note that ρ is trivial on $G_{x_2,\frac{1}{2}+}$, i.e., factors through $K/G_{x_2,\frac{1}{2}+}$.

Fact 3.4.1. *The representation* $\text{c-ind}_K^{\mathrm{SL}_2(F)} \rho$ *is a supercuspidal irreducible representation of depth* $\frac{1}{2}$.

If $p \neq 2$, this is a very special case of Yu's construction as we will see below. This construction also works for $p = 2$ and is an example of a *simple supercuspidal representation* introduced by Gross and Reeder ([GR10, §9.3]), which in turn are special cases of *epipelagic representations* as introduced by Reeder and Yu ([RY14]), which are representations of smallest positive depth. More precisely, for $x \in \mathscr{B}(G, F)$, let $r(x)$ be the smallest positive real number for which $G_{x,r(x)} \neq G_{x,r(x)+}$. Then an irreducible representation (π, V) is called *epipelagic* if there exists $x \in \mathscr{B}(G, F)$ such that $V^{G_{x,r(x)+}}$ is non-trivial and (π, V) has depth $r(x)$ ([RY14, §2.5]).

3.5 Generic characters

In order to generalize the example of the previous subsection and to eventually present Yu's general construction of supercuspidal representations, we need the notion of twisted Levi subgroups and generic characters.

Definition 3.5.1. A subgroup G' of G is a *twisted Levi subgroup* if G'_E is a Levi subgroup of G_E for some finite field extension E over F.

If G' is a twisted Levi subgroup of G, and we assume that G' splits over a tamely ramified field extension of F, then we have an embedding of the enlarged Bruhat–Tits building $\widetilde{\mathscr{B}}(G', F)$ of G' into the enlarged Bruhat–Tits building $\widetilde{\mathscr{B}}(G, F)$ of G. This embedding is unique up to translation by $X_*(Z(G')) \otimes_{\mathbb{Z}} \mathbb{R}$, and its image is independent of the embedding. We will fix such embeddings when working with twisted Levi subgroups to view $\widetilde{\mathscr{B}}(G', F)$ as a subset of $\widetilde{\mathscr{B}}(G, F)$.

In order to define generic characters (following [Fin22, §2.1], which is based on [Yu01, §9], but is slightly more general for small primes, see [Fin22, Remark 2.2] for details), we first define the notion of generic elements in the dual of the Lie algebra and then use the Moy–Prasad isomorphism to obtain the notion of generic characters.

We denote by $\Phi(G, T)$ the absolute root system of G with respect to T, i.e., the roots of $G_{\bar{F}}$ with respect to $T_{\bar{F}}$, where $\bar{F}$ denotes a separable closure of F. We also extend the valuation val on F to a valuation $\mathrm{val} : \bar{F} \to \mathbb{Q} \cup \{\infty\}$ on $\bar{F}$ and denote by $\mathcal{O}_{\bar{F}}$ all the elements of $\bar{F}$ with non-negative or infinite valuation.

Let $G' \subseteq G$ be a twisted Levi subgroup that splits over a tamely ramified field extension of F, and denote by $(\mathrm{Lie}^*(G'))^{G'}$ the subscheme of the linear dual of the Lie algebra $\mathrm{Lie}(G')$ of G' fixed by (the dual of) the adjoint action of G'.

Definition 3.5.2. Let $x \in \widetilde{\mathscr{B}}(G', F)$ and $r \in \mathbb{R}_{>0}$.

(a) An element X of $(\mathrm{Lie}^*(G'))^{G'}(F) \subset \mathrm{Lie}^*(G')(F)$ is called *G-generic of depth r* (or *(G, G')-generic of depth r*) if the following three conditions hold.

(GE0) For some (equivalently, every) point $x \in \widetilde{\mathscr{B}}(G', F)$, we have $X \in \mathrm{Lie}^*(G')_{x,r} \smallsetminus \mathrm{Lie}^*(G')_{x,r+}$.

(GE1) $\mathrm{val}(X(H_\alpha)) = r$ for all $\alpha \in \Phi(G,T) \smallsetminus \Phi(G',T)$ for some (equivalently, every) maximal torus T of G', where $H_\alpha := d\check{\alpha}(1) \in \mathfrak{g}(\bar{F})$ with $d\check{\alpha}$ the derivative of the coroot $\check{\alpha} \in X_*(T_{\bar{F}})$ of α.

(GE2) **GE2** of [Yu01, §8] holds, which we recall below and which is implied by (GE1) if p does not divide the order of the absolute Weyl group of G.

(b) A character ϕ of $G'(F)$ is called *G-generic (or (G, G')-generic) relative to x of depth* r if ϕ is trivial on $G'_{x,r+}$ and the restriction of ϕ to $G'_{x,r}/G'_{x,r+} \simeq \mathfrak{g}'_{x,r}/\mathfrak{g}'_{x,r+}$ is given by $\varphi \circ X$ for some element $X \in (\mathrm{Lie}^*(G'))^{G'}(F)$ that is (G, G')-generic of depth $-r$.

The equivalence in (GE0) is proven in [Fin22, Lemma 2.3].

In order to explain Condition (GE2), let $X \in (\mathrm{Lie}^*(G'))^{G'}(F) \subset \mathrm{Lie}^*(G')(F)$ satisfy (GE0) and (GE1) for a maximal torus T of G'. We denote by $X_{\mathfrak{t}}$ the restriction of X to $\mathfrak{t}(\bar{F})$ and choose an element ϖ_r of valuation r in $\bar{F}$. Then, under the identification of $\mathfrak{t}^*(\bar{F})$ with $X^*(T_{\bar{F}}) \otimes_{\mathbb{Z}} \bar{F}$, the element $\frac{1}{\varpi_r} X_{\mathfrak{t}}$ is contained in $X^*(T_{\bar{F}}) \otimes_{\mathbb{Z}} \mathcal{O}_{\bar{F}}$, and we denote its image under the surjection $X^*(T_{\bar{F}}) \otimes_{\mathbb{Z}} \mathcal{O}_{\bar{F}} \twoheadrightarrow X^*(T_{\bar{F}}) \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_q$ by $\bar{X}_{\mathfrak{t}}$. Now we can state Condition (GE2):

(GE2) The subgroup of the absolute Weyl group of G that fixes $\bar{X}_{\mathfrak{t}}$ is the absolute Weyl group of G'.

Remark 3.5.3. By [Yu01, Lemma 8.1], Condition **(GE1)** implies **(GE2)** if p is not a torsion prime for the dual root datum of G, i.e., in particular, if p does not divide the order of the absolute Weyl group of G.

Remark 3.5.4. It is work in progress to construct supercuspidal representations for a more general notion of "generic" that does not require (GE2) to be satisfied (and only requires a weaker version of (GE1)).

Remark 3.5.5. If a character ϕ of $G'(F)$ is (G, G')-generic relative to x of depth r, then it is also (G, G')-generic relative to x' of depth r for every $x' \in \widetilde{\mathscr{B}}(G', F)$, i.e., the notion of genericity does not depend on the choice of point x ([AFMOb, Lemma 3.3.1]).

Remark 3.5.6. We caution the reader that an element in $\mathrm{Lie}^*(G')(F)$ that is G-generic of depth r is sometimes called "G-generic of depth $-r$" in the literature (e.g., in [Yu01, §8] and [Fin22, §2.1]). However, such an element has depth r, in the sense of it being contained in $\mathrm{Lie}^*(G')_{x,r} \smallsetminus \mathrm{Lie}^*(G')_{x,r+}$, and therefore the latter convention has led to some confusion in the literature in the past.

Remark 3.5.7. Usually the notion of "(G, G')-generic" is only defined for $G' \subsetneq G$. However, sometimes it is convenient to also consider the case $G' = G$, see, e.g., [AFMOb], and in this case our definition implies that a (G,G)-generic character of depth r has indeed depth r. In particular, we do not consider the trivial character a (G,G)-generic character of depth r. This differs from Yu's convention in [Yu01, § 15, p. 616] where he considers trivial characters as G-generic of depth r. We have chosen the above more restrictive definition of (G,G)-generic characters of depth r as it allows to construct more uniformly representations of depth r from a (G,G')-generic character of depth r without having to treat the case $G = G'$ separately.

To provide some examples of generic characters, we consider the case that $F = \mathbb{Q}_7$, $G = \mathrm{GL}_2$ and G' is the diagonal torus $T \subset \mathrm{GL}_2$. We let $\psi : \mathbb{Q}_7^\times \to \mathbb{C}^\times$ be a character of depth 1. Then the following three characters

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \psi(t_1) \quad \text{and} \quad \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \psi(t_2) \quad \text{and} \quad \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \psi(t_1 t_2^{-1})$$

are (G,T)-generic of depth 1 relative to any point $x \in \widetilde{\mathscr{B}}(T,\mathbb{Q}_7) \subset \widetilde{\mathscr{B}}(G,\mathbb{Q}_7)$. The two characters

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \psi(t_1 t_2) \quad \text{and} \quad \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \psi(t_1 t_2^{-6})$$

are also of depth 1 relative to any point $x \in \widetilde{\mathscr{B}}(T,\mathbb{Q}_7) \subset \widetilde{\mathscr{B}}(G,\mathbb{Q}_7)$, but they are not (G,T)-generic of depth 1.

3.6 More examples of positive-depth supercuspidal representations

We will now use generic characters to provide a construction of supercuspidal representations of positive depth that generalizes the example provided in Section 3.4 and have arbitrary large depth. As input for the construction we take the following data

(a) $S \subset G$ an elliptic maximal torus of G that splits over tamely ramified extension E of F,

(b) $x \in \widetilde{\mathscr{B}}(S,F) \subset \widetilde{\mathscr{B}}(G,F)$,

(c) $r \in \mathbb{R}_{>0}$ such that $G_{x,\frac{r}{2}} = G_{x,\frac{r}{2}+}$,

(d) $\phi : S(F) \to \mathbb{C}^\times$ a character that is (G,S)-generic relative to x of depth r.

The supercuspidal representation that we construct from this input is of the form $\text{c-ind}_K^{G(F)} \hat{\phi}$ with $K = S(F)G_{x,\frac{r}{2}}$ and $\hat{\phi}$ the extension of ϕ obtained by "sending the root groups to 1". More precisely, $\hat{\phi}$ is the unique character of $S(F)G_{x,\frac{r}{2}}$ that satisfies

(i) $\hat{\phi}|_{S(F)} = \phi$, and

(ii) $\hat{\phi}|_{G_{x,\frac{r}{2}}}$ factors through

$$\begin{aligned} G_{x,\frac{r}{2}} = G_{x,\frac{r}{2}+} &\twoheadrightarrow G_{x,\frac{r}{2}+}/G_{x,r+} \simeq \mathfrak{g}_{x,\frac{r}{2}+}/\mathfrak{g}_{x,r+} = (\mathfrak{s}(F)\oplus\mathfrak{r}(F))_{x,\frac{r}{2}+}/(\mathfrak{s}(F)\oplus\mathfrak{r}(F))_{x,r+} \\ &\twoheadrightarrow \mathfrak{s}_{x,\frac{r}{2}+}/\mathfrak{s}_{x,r+} \simeq S_{x,\frac{r}{2}+}/S_{x,r+}, \end{aligned}$$

on which it is induced by $\phi|_{S_{x,\frac{r}{2}+}}$, where the subspace $\mathfrak{r}(F)$ of root subspaces is defined to be

$$\mathfrak{r}(F) = \mathfrak{g}(F) \cap \bigoplus_{\alpha\in\Phi(G,S)} \mathfrak{g}(E)_\alpha,$$

and the surjection $\mathfrak{s}(F)\oplus\mathfrak{r}(F) \twoheadrightarrow \mathfrak{s}(F)$ sends $\mathfrak{r}(F)$ to zero. The isomorphisms used are the Moy–Prasad isomorphisms from Fact 2.2.2(e).

Fact 3.6.1. *The representation* $\text{c-ind}_{S(F)G_{x,\frac{r}{2}}}^{G(F)} \hat{\phi}$ *is a supercuspidal irreducible representation of depth r.*

The construction of these representations is a special case of the construction of supercuspidal representations provided by Adler ([Adl98]) that was later generalized by Yu ([Yu01]). (These references impose a condition on p, but this is not necessary for the above special case.)

We recover the representation constructed in Section 3.4 under the assumption that $p \neq 2$ from the following input

(a) $S \subset \mathrm{SL}_2$ is the torus that satisfies for every field extension F' of F

$$S(F') = \left\{ \begin{pmatrix} a & b \\ \varpi b & a \end{pmatrix} \in \mathrm{SL}_2(F') \,|\, a, b \in F' \right\}.$$

Then S splits over the quadratic extension $F(\sqrt{\varpi})$ of F.

(b) The Bruhat–Tits building $\mathscr{B}(\mathrm{SL}_2, F) = \widetilde{\mathscr{B}}(\mathrm{SL}_2, F)$ is an infinite tree of valency $|\mathbb{F}_q| + 1$ and the Bruhat–Tits building $\mathscr{B}(S, F) = \widetilde{\mathscr{B}}(S, F)$ of S is a single point that embeds into $\mathscr{B}(\mathrm{SL}_2, \mathbb{Q}_p)$ as the barycenter x of an edge, see Figure 2. Hence there is a unique choice for $x \in \widetilde{\mathscr{B}}(S, F) \subset \widetilde{\mathscr{B}}(G, F)$.

(c) We let $r = \frac{1}{2}$.

(d) We define $\phi : S(F) \to \mathbb{C}^\times$ by

$$\phi\left(\begin{pmatrix} a & b \\ \varpi b & a \end{pmatrix}\right) = \varphi(2b).$$

Then ϕ is (SL_2, S)-generic of depth $\frac{1}{2}$.

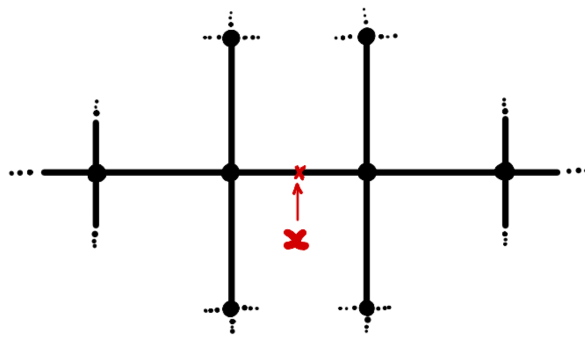

Figure 2: Excerpt of the Bruhat–Tits building $\mathscr{B}(\mathrm{SL}_2, \mathbb{Q}_3)$

Remark 3.6.2. Since S is an elliptic maximal torus of G, the building $\widetilde{\mathscr{B}}(S, F)$ is equal to $x + X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} = x + X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$, where we recall that $X_*(?) = \mathrm{Hom}_F(\mathbb{G}_m, ?)$, and hence the choice of $x \in \widetilde{\mathscr{B}}(S, F)$ has no influence on the construction. Moreover, the real number r is just the depth of ϕ, i.e., can be read off from ϕ. Thus, the actual input for the above construction consists only of the pair (S, ϕ).

We will now generalize this construction to allow the case $G_{x,\frac{r}{2}} \neq G_{x,\frac{r}{2}+}$, which Yu has dealt with using the theory of Heisenberg–Weil representations and which is why he assumes $p \neq 2$, and to allow a more general sequence of twisted Levi subgroups instead of only $S \subset G$.

3.7 The input for the construction by Yu

We assume from now on that $p \neq 2$. For a generalization of the below construction of supercuspidal representations that also works if $p = 2$ we refer the reader to [FS].

The input for the construction of supercuspidal representations by Yu (following the notation of [Fin21a]) is a tuple $((G_i)_{1 \leq i \leq n+1}, x, (r_i)_{1 \leq i \leq n}, \rho^0, (\phi_i)_{1 \leq i \leq n})$ for some non-negative integer n where

(a) $G = G_1 \supseteq G_2 \supsetneq G_3 \supsetneq \ldots \supsetneq G_{n+1}$ are twisted Levi subgroups of G that split over a tamely ramified extension of F,

(b) $x \in \widetilde{\mathscr{B}}(G_{n+1}, F) \subset \widetilde{\mathscr{B}}(G, F)$,

(c) $r_1 > r_2 > \ldots > r_n > 0$ are real numbers,

(d) ϕ_i, for $1 \leq i \leq n$, is a character (i.e., a one-dimensional representation) of $G_{i+1}(F)$ of depth r_i,

(e) ρ^0 is an irreducible representation of $(G_{n+1})_{[x]}$ that is trivial on $(G_{n+1})_{x,0+}$,

satisfying the following conditions

(i) $Z(G_{n+1})/Z(G)$ is anisotropic, i.e., its F-points are a compact group,

(ii) the image $[x]$ of the point x in $\mathscr{B}(G_{n+1}, F)$ is a vertex, i.e., a polysimplex of minimal dimension,

(iii) ϕ_i is (G_i, G_{i+1})-generic relative to x of depth r_i for all $1 \leq i \leq n$,

(iv) $\rho^0|_{(G_{n+1})_{x,0}}$ is a cuspidal representation of the reductive group $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$.

We will call a tuple satisfying the above conditions a *supercuspidal datum*.

Aside 3.7.1. Our conventions for the notation (following [Fin21a]) differ slightly from those in [Yu01]. In particular, Yu's notation for the twisted Levi sequence is $G^0 \subsetneq G^1 \subsetneq G^2 \subsetneq \ldots \subsetneq G^d$. The reader can find a translation between the two different notations in [Fin21a, Remark 2.4] and in Remark 4.6.1 for the reverse direction.

Example of a supercuspidal datum. We provide an example of a supercuspidal datum for the group $G = \mathrm{SL}_2$ with p an odd prime. We let $n = 1$.

(a) We have $G_1 = G$ and let $G_2 = S$ be the non-split torus $S \subset \mathrm{SL}_2$ that satisfies

$$S(F') = \left\{ \begin{pmatrix} a & b \\ \varpi b & a \end{pmatrix} \in \mathrm{SL}_2(F') \,|\, a, b \in F' \right\} \text{ for all field extensions } F' \text{ of } F.$$

(b) The point x is the unique point of $\widetilde{\mathscr{B}}(S, F) \subset \widetilde{\mathscr{B}}(G, F)$.

(c) We let $r_1 = \frac{1}{2}$.

(d) We define $\phi_1 : S(F) \to \mathbb{C}^\times$ by $\phi_1 \left(\begin{pmatrix} a & b \\ \varpi b & a \end{pmatrix} \right) = \varphi(2b)$.

(e) $S_{[x]} = S(F) = \{\pm \mathrm{Id}\} \times S_{x,0+}$ and we let ρ^0 be the trivial representation on a one-dimensional vector space.

The supercuspidal representation constructed from this supercuspidal datum following the recipe in the next subsection turns out to be the representation described in Section 3.4.

3.8 The construction of supercuspidal representations à la Yu

In this section we outline how Yu ([Yu01]) constructs from a supercuspidal datum

$$((G_i)_{1 \leq i \leq n+1}, x, (r_i)_{1 \leq i \leq n}, \rho^0, (\phi_i)_{1 \leq i \leq n})$$

a compact-mod-center open subgroup $\widetilde{K}$ and a representation $\widetilde{\rho}$ of $\widetilde{K}$ such that $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ is an irreducible supercuspidal representation of $G(F)$.

The compact-mod-center open subgroup $\widetilde{K}$ is given by

$$\widetilde{K} = (G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}} \dots (G_n)_{x,\frac{r_n}{2}}(G_{n+1})_{[x]},$$

where $(G_{n+1})_{[x]}$ denotes the stabilizer in $G_{n+1}(F)$ of the point $[x]$ in the (reduced) Bruhat–Tits building $\mathscr{B}(G_{n+1}, F)$.

The representation $\widetilde{\rho}$ is a tensor product of two representations ρ^0 and κ^{nt},

$$\widetilde{\rho} = \rho^0 \otimes \kappa^{\mathrm{nt}},$$

where ρ^0 also denotes the extension of the representation ρ^0 of $(G_{n+1})_{[x]}$ to $\widetilde{K}$ that is trivial on $(G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}} \dots (G_n)_{x,\frac{r_n}{2}}$. The representation κ^{nt} is built out of the characters $\phi_1, \dots, \phi_n$. If $n = 0$, then κ^{nt} is trivial and we are in the setting of depth-zero representations.

Remark 3.8.1. In the literature κ^{nt} is often denoted by κ. We have chosen to add the superscript "nt" to indicate that this is the original not-twisted construction and therefore better align our notation with [AFMOb], which is a crucial source for the AWS projects. We will later twist the representation κ^{nt} to obtain a twisted version that we will then call κ, see Section 3.10 and 4.6 for details. Similarly, ρ^0 is in the literature on constructions of supercuspidal representations usually simply denoted by ρ, however the symbol ρ will play a different role in the construction of types below. Therefore we have chosen to add the superscript "0" for the depth-zero representation ρ^0 already here, which leads to a notation that is consistent with the one used in Section 4.6 and with [AFMOb].

We will first sketch the construction of κ^{nt} in the case $n = 1$, i.e., when the supercuspidal datum is of the form $((G = G_1 \supset G_2 = G_{n+1}), x, (r_1), \rho^0, (\phi_1))$. To simplify notation, we write $r = r_1$ and $\phi = \phi_1$, and we assume $G_1 \neq G_2$. In this case $\widetilde{K} = (G_1)_{x,\frac{r}{2}}(G_2)_{[x]}$.

Step 1 (extending the character ϕ as far as possible): The first step consists of extending the character ϕ to a character $\hat{\phi}$ of $(G_1)_{x,\frac{r}{2}+}(G_2)_{[x]}$. This is done as in Section 3.6 by sending the root groups outside G_2 to 1. More precisely, $\hat{\phi}$ is the unique character of $(G_1)_{x,\frac{r}{2}+}(G_2)_{[x]}$ that satisfies

- $\hat{\phi}|_{(G_2)_{[x]}} = \phi$, and
- $\hat{\phi}|_{(G_1)_{x,\frac{r}{2}+}}$ factors through

$$\begin{aligned}(G_1)_{x,\frac{r}{2}+}/(G_1)_{x,r+} &\simeq \mathfrak{g}_{x,\frac{r}{2}+}/\mathfrak{g}_{x,r+} = (\mathfrak{g}_2(F) \oplus \mathfrak{r}(F))_{x,\frac{r}{2}+}/(\mathfrak{g}_2(F) \oplus \mathfrak{r}(F))_{x,r+} \\ &\to (\mathfrak{g}_2)_{x,\frac{r}{2}+}/(\mathfrak{g}_2)_{x,r+} \simeq (G_2)_{x,\frac{r}{2}+}/(G_2)_{x,r+},\end{aligned}$$

  on which it is induced by $\phi|_{(G_2)_{x,\frac{r}{2}+}}$, where we use the Moy–Prasad isomorphism, $\mathfrak{r}(F)$ is defined to be

$$\mathfrak{r}(F) = \mathfrak{g}(F) \cap \bigoplus_{\alpha \in \Phi(G_E, T_E) \smallsetminus \Phi((G_2)_E, T_E)} \mathfrak{g}(E)_\alpha$$

  for some maximal torus T of G_2 that splits over a tamely ramified extension E of F with $x \in \widetilde{\mathscr{A}}(T_E, E)$, and the surjection $\mathfrak{g}_2(F) \oplus \mathfrak{r}(F) \twoheadrightarrow \mathfrak{g}_2(F)$ sends $\mathfrak{r}(F)$ to zero.

Step 2 (Heisenberg representation): As second step we extend the (one-dimensional) representation $\hat{\phi}|_{(G_1)_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}}}$ to a representation (ω, V_ω) of $(G_1)_{x,\frac{r}{2}}$. We write $V_{\frac{r}{2}}$ for the quotient

$$V_{\frac{r}{2}} = (G_1)_{x,\frac{r}{2}}/((G_1)_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}})$$

and we view $V_{\frac{r}{2}}$ as an $\mathbb{F}_p$-vector space. (It can also be viewed as an $\mathbb{F}_q$-vector space, but here we only consider the underlying $\mathbb{F}_p$-vector space structure.) Then one can show that the pairing

$$\langle g, h\rangle := \hat{\phi}(ghg^{-1}h^{-1}), \; g, h \in (G_1)_{x,\frac{r}{2}}$$

defines a non-degenerate symplectic form on $V_{\frac{r}{2}} = (G_1)_{x,\frac{r}{2}}/((G_1)_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}})$ when we choose an identification between the p-th roots of unity in $\mathbb{C}^\times$ and $\mathbb{F}_p$.

Now the theory of Heisenberg representations implies that there exists a unique irreducible representation (ω, V_ω) of $(G_1)_{x,\frac{r}{2}}$ that restricted to $(G_1)_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}}$ acts via $\hat{\phi}$ (times identity), and the dimension of V_ω is $\sqrt{\# V_{\frac{r}{2}}} = p^{(\dim_{\mathbb{F}_p} V_{\frac{r}{2}})/2}$.

Step 3 (Weil representation): The final step of the construction consists of extending the action of $(G_1)_{x,\frac{r}{2}}$ on V_ω via ω to an action of $\widetilde{K} = (G_1)_{x,\frac{r}{2}}(G_2)_{[x]}$ on V_ω by defining an action of $(G_2)_{[x]}$ on V_ω that is compatible with ω. In order to obtain this action, we first observe that $(G_2)_{[x]}$ acts on $V_{\frac{r}{2}}$ via conjugation and that this action preserves the symplectic form $\langle \cdot, \cdot \rangle$. This provides a morphism from $(G_2)_{[x]}$ to the group $\mathrm{Sp}(V_{\frac{r}{2}})$ of symplectic isomorphisms of $V_{\frac{r}{2}}$. Now the Weil representation is a representation of the symplectic group $\mathrm{Sp}(V_{\frac{r}{2}})$ on the space V_ω of the Heisenberg representation of the symplectic vector space that is compatible with the Heisenberg representation in the following sense. Using the composition of the morphism $(G_2)_{[x]} \to \mathrm{Sp}(V_{\frac{r}{2}})$ with the Weil representation tensored with the character ϕ allows us to extend the representation (ω, V_ω) from $(G_1)_{x,\frac{r}{2}}$ to $(G_1)_{x,\frac{r}{2}}(G_2)_{[x]}$. We denote the resulting representation of $\widetilde{K} = (G_1)_{x,\frac{r}{2}}(G_2)_{[x]}$ also by (ω, V_ω) and set $(\kappa^{\mathrm{nt}}, V_\kappa^{\mathrm{nt}}) = (\omega, V_\omega)$.

This concludes the construction of κ^{nt} and hence $\widetilde{\rho} = \rho^0 \otimes \kappa^{\mathrm{nt}}$ in the case of $n = 1$. For a more general supercuspidal datum $((G_i)_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, \rho^0, (\phi_i)_{1\leq i\leq n})$ with $n > 1$ we construct from each character ϕ_i $(1 \leq i \leq n)$ a representation (ω_i, V_{ω_i}) analogous to the construction of (ω, V_ω) above. Then we define κ^{nt} to be the tensor product of all those representations, i.e.

$$(\kappa^{\mathrm{nt}}, V_\kappa^{\mathrm{nt}}) = \left(\bigotimes_{1\leq i\leq n} \omega_i, \bigotimes_{1\leq i\leq n} V_{\omega_i} \right).$$

For the details we refer the reader to [Fin21a, §2.5], which is based on [Yu01].

Theorem 3.8.2 ([Yu01, Fin21a])**.** *The representation* $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ *is a supercuspidal smooth irreducible representation of* $G(F)$.

We will sketch the structure of the proof in the next subsection.

3.9 Sketch of the proof that the representations are supercuspidal

In order to prove that $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ is supercuspidal it suffices to prove that it is irreducible by Lemma 3.2.1. First one notes that $\widetilde{\rho}$ itself is irreducible. We assume that an element $g \in G(F)$ intertwines $\widetilde{\rho}$. Now the main task is to show that $g \in \widetilde{K}$ so that we can apply Lemma 3.2.3. This is done in two steps.

Step 1. We show recursively that $g \in \widetilde{K} G_{n+1} \widetilde{K}$ using that the characters ϕ_i are generic.

The key part for this step is [Yu01, Theorem 9.4], which in the example of $n = 1$ spelled out above implies the following lemma.

Lemma 3.9.1 ([Yu01]). *Suppose that g intertwines $\hat{\phi}|_{(G_1)_{x,\frac{r}{2}+}}$. Then $g \in (G_1)_{x,\frac{r}{2}} G_2(F)(G_1)_{x,\frac{r}{2}}$.*

As mentioned above, this lemma crucially uses the fact that ϕ is (G, G_2)-generic relative to x of depth r (if $G_1 \neq G_2$) and we refer to [Yu01, Theorem 9.4] for the proof.

Step 2. By Step 1 we may assume that $g \in G_{n+1}(F)$. Step 2 consists of showing that then $g \in (G_{n+1})_{[x]}$ using the structure of the Heisenberg–Weil representation and that $\rho^0|_{(G_{n+1})_{x,0}}$ is cuspidal. The spirit of this step is similar to the proof of Theorem 3.3.1, but in this more general setting it additionally requires an intricate study of the involved Heisenberg–Weil representations.

The reader interested in the full details of the proof is encouraged to read [Fin21a, §3], which is only about four pages long and refers to precise statements in [Yu01] that allow an easy backtracking within [Yu01] if the reader is interested in all the details that make the complete proof.

3.10 A twist of Yu's construction

Let $((G_i)_{1 \leq i \leq n+1}, x, (r_i)_{1 \leq i \leq n}, \rho^0, (\phi_i)_{1 \leq i \leq n})$ be a supercuspidal datum. Instead of associating to this supercuspidal datum the representation $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ constructed by Yu, a new suggestion by Fintzen, Kaletha and Spice ([FKS23]) consists of associating the representation $\text{c-ind}_{\widetilde{K}}^{G(F)}(\epsilon\widetilde{\rho})$ for an explicitly constructed character $\epsilon : \widetilde{K} \to \{\pm 1\}$. We refer the reader to [FKS23, p. 2259] for the definition of ϵ as it is rather involved. There are multiple reasons for the introduction of this quadratic twist in the parametrization. For example, it restores the validity of Yu's original proof ([Yu01]) that $\text{c-ind}_{\widetilde{K}}^{G(F)}(\epsilon\widetilde{\rho})$ is a supercuspidal irreducible representation, which is not valid for the non-twisted version as it relied on a misprinted statement in [Gér77]. In particular, we restore the validity of the intertwining results [Yu01, Proposition 14.1 and Theorem 14.2] for the twisted construction that form the heart of Yu's proof. Instead of stating the results in full generality, which would involve introducing additional notation, we state its implication in the setting that we already introduced above.

Proposition 3.10.1 ([Yu01,FKS23]). *Let $((G = G_1 \supsetneq G_2 = G_{n+1}), x, (r_1 = r), \rho^0, (\phi_1 = \phi))$ be a supercuspidal datum from which we construct a representation κ^{nt} of $\widetilde{K} = (G_1)_{x,\frac{r}{2}}(G_2)_{[x]}$ as in Section 3.8. Set $\kappa = \epsilon\kappa^{\text{nt}}$. Then for $g \in G_2(F)$, we have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\widetilde{K} \cap g\widetilde{K}g^{-1}}(\kappa, {}^g(\kappa)) = 1.$$

This result also holds in a more general setting in which we drop the assumption that $Z(G_2)/Z(G)$ is anisotropic. We refer the reader to [FKS23, Corollary 4.1.11 and Corollary 4.1.12] for the detailed statements and proofs.

Applications of the existence of the above quadratic character $\epsilon : \widetilde{K} \to \{\pm 1\}$ include being able (under some assumptions on F) to provide a character formula for the supercuspidal representations $\text{c-ind}_{\widetilde{K}}^{G(F)}(\epsilon\widetilde{\rho})$ ([Spi18, Spi, FKS23]), to suggest a local Langlands correspondence for all supercuspidal Langlands parameters ([Kal]) and to prove the stability and many instances of the endoscopic character identities for the resulting supercuspidal L-packets that such a local Langlands correspondence is predicted to satisfy ([FKS23]).

3.11 Exhaustiveness of the construction of supercuspidal representations

Theorem 3.11.1 ([Kim07, Fin21d]). *Suppose that G splits over a tamely ramified field extension of F and that p does not divide the order of the absolute Weyl group of G. Then every supercuspidal smooth irreducible representation of $G(F)$ arises from Yu's construction, i.e., via Theorem 3.8.2.*

This result was shown by Kim ([Kim07]) in 2007 under the additional assumptions that F has characteristic zero and that p is "very large". Her approach was very different from the recent approach in [Fin21d]. Kim proves statements about a measure one subset of all smooth irreducible representations of $G(F)$ by matching summands of the Plancherel formula for the group and the Lie algebra, while the recent approach in [Fin21d] is more explicit and can be used to recursively exhibit a supercuspidal datum for the construction of the given representation. The latter approach consists of two main steps. The first step is to prove that every supercuspidal smooth irreducible representation of $G(F)$ contains a (maximal) datum as defined in [Fin21d], which can be viewed as a skeleton of a supercuspidal datum. The second step consists of obtaining a supercuspidal datum from that maximal datum and showing that the representation we started with is isomorphic to the one constructed from this supercuspidal datum. We refer the reader to [Fin21d] for the details and to Section 5 of [Fin23] for an expanded overview.

4 Bernstein blocks, types and Hecke algebras

We have seen (Fact 1.2.8) that every irreducible representation embeds into the parabolic induction of a supercuspidal representation of a Levi subgroup of G. Thus the supercuspidal representations form the buildings blocks, and the previous section was concerned with constructing those building blocks. In this section we want to now understand the whole category of all smooth representations of $G(F)$.

4.1 Bernstein decomposition

Let (π, V) be an irreducible smooth representation of $G(F)$. By Fact 1.2.8 there exists a parabolic subgroup $P \subseteq G$ with Levi subgroup M and a supercuspidal representation

(σ, W) of $M(F)$ such that $\pi \subseteq \mathrm{Ind}_{P(F)}^{G(F)} \sigma$. If $P' \subseteq G$ is another parabolic subgroup with Levi subgroup M' and a supercuspidal representation (σ', W') of $M'(F)$ such that $\pi \subseteq \mathrm{Ind}_{P'(F)}^{G'(F)} \sigma'$, then it turns out that there exists $g \in G(F)$ such that $M' = gMg^{-1}$ and $\sigma' \simeq {}^g\sigma$. We call the $G(F)$-conjugacy class of the pair $(M, (\sigma, W))$ the *supercuspidal support* of (π, V).

In order to decompose the category of all smooth representations we need to define a weaker equivalence class on the pairs consisting of Levi subgroups and supercuspidal representations.

Definition 4.1.1. A smooth character $\chi : G(F) \to \mathbb{C}^*$ is called an *unramified character* if the restriction of χ to any compact subgroup of $G(F)$ is trivial.

Definition 4.1.2. Let M and M' be Levi subgroups of (parabolic subgroups of) G and let σ and σ' be supercuspidal representations of $M(F)$ and $M'(F)$, respectively. We say that (M, σ) and (M', σ') are *inertially equivalent* if and only if there exist $g \in G(F)$ and an unramified character χ of $M'(F)$ such that $M' = gMg^{-1}$ and $\sigma' \simeq {}^g\sigma \otimes \chi$.

We denote the inertial equivalence by $\sim$, write $[M, \sigma]_G$ for the inertial equivalence class of the pair (M, σ), and denote by $\mathfrak{I}(G)$ the set of inertial equivalence classes G, i.e., $\mathfrak{I}(G) = \{[M, \sigma]_G\}$ where M runs over the Levi subgroups of G and σ is a supercuspidal representation of $M(F)$. We might simply write $[M, \sigma]$ instead of $[M, \sigma]_G$ if the group G is clear from the context.

Let $[M, \sigma] \in \mathfrak{I}(G)$. Then we denote by $\mathrm{Rep}(G)_{[M,\sigma]}$ the full subcategory of the category of smooth representations $\mathrm{Rep}(G)$ of $G(F)$ whose objects are the following: A smooth representation π of $G(F)$ is contained in $\mathrm{Rep}(G)_{[M,\sigma]}$ if and only if for every irreducible subquotient π' of π, there exists a parabolic subgroup P' with Levi subgroup M' and a supercuspidal representation σ' of $M'(F)$ with $(M', \sigma') \in [M, \sigma]$ such that $\pi' \hookrightarrow \mathrm{Ind}_{P'(F)}^{G'(F)} \sigma'$.

Theorem 4.1.3 ([Ber84])**.** *We have an equivalence of categories*

$$\mathrm{Rep}(G) \simeq \prod_{[M,\sigma] \in \mathfrak{I}(G)} \mathrm{Rep}(G)_{[M,\sigma]},$$

and each full subcategory $\mathrm{Rep}(G)_{[M,\sigma]}$ *is indecomposable.*

The above equivalence of categories is called the *Bernstein decomposition* and the full subcategory $\mathrm{Rep}(G)_{[M,\sigma]}$ is called a *Bernstein block*.

4.2 Types and covers

The structure of the Bernstein blocks can be analyzed via type theory that was introduced by Bushnell and Kutzko ([BK98]).

Definition 4.2.1. Let $[M, \sigma] \in \mathfrak{I}(G)$. A pair (K, ρ) consisting of a compact, open subgroup K of $G(F)$ and an irreducible smooth representation ρ of K is an $[M, \sigma]$-*type* if the following property holds: For every irreducible smooth representation π of $G(F)$ the following are equivalent:

(i) π is an object in $\mathrm{Rep}(G)_{[M,\sigma]}$,

(ii) ρ is a subrepresentation of the restriction $\pi|_K$ of π to K (i.e., $\mathrm{Hom}_K(\rho, \pi) \neq \{0\}$).

The notion of $[M, \sigma]$-types for $M = G$ is closely related to the construction of supercuspidal representations, as the following result shows.

Fact 4.2.2 ([BK98, (5.4)]). *Let $\widetilde{K}$ be a compact-mod-center, open subgroup of $G(F)$, and let $(\widetilde{\rho}, V_{\widetilde{\rho}})$ be an irreducible smooth representation of $\widetilde{K}$ such that $\pi := \text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ is irreducible, hence supercuspidal. Let K be the (unique) maximal compact subgroup of $\widetilde{K}$, and let ρ be an irreducible component of the restriction of $\widetilde{\rho}$ to K. Then (K, ρ) is a $[G, \pi]$-type.*

Thus our construction of supercuspidal representations above leads to a plethora of examples of types. This construction can also be generalized to obtain types for other Bernstein blocks using the theory of covers as follows.

Definition 4.2.3. Let M be a Levi subgroup of G, let K be a compact, open subgroup of $G(F)$, and let K_M be a compact, open subgroup of $M(F)$. Let (ρ, V_ρ) and (ρ_M, V_{ρ_M}) be irreducible smooth representations of K and of K_M, respectively.
The pair (K, ρ) is a *G-cover* of (K_M, ρ_M) if for every parabolic subgroup $P \subset G$ with Levi decomposition $P = MN$ the following properties hold.

(i) $K = (K \cap N) \cdot (K \cap M) \cdot (K \cap \overline{N})$ and $K \cap M = K_M$ where $\overline{P} = M\overline{N}$ denotes the opposite parabolic subgroup of G with respect to M (i.e., $P \cap \overline{P} = M$).

(ii) $\rho|_{K_M} = \rho_M$ and $\rho|_{K \cap N} = 1_{V_\rho}$ and $\rho|_{K \cap \overline{N}} = 1_{V_\rho}$.

(iii) For every irreducible smooth representation (π, V) of $G(F)$, the restriction of the surjection $V \twoheadrightarrow V_N := V / \langle v - \pi(n)(v) \,|\, n \in N(F) \rangle$ to the subspace $V^{(K,\rho)}$ is injective, where $V^{(K,\rho)}$ denotes the largest subspace of V on which the restriction of π to K is isomorphic to a direct sum of copies of ρ.

Fact 4.2.4 ([BK98]). *Let (K_M, ρ_M) be an $[M, \sigma]_M$-type in M, and let (K, ρ) be a G-cover of (K_M, ρ_M). Then (K, ρ) is an $[M, \sigma]_G$-type in G.*

Using this theorem and results of [MP94, MP96], one can check that if G is a split reductive group with split maximal torus T and $\mathrm{Iw} = G_{x,0} \subset G(F)$ for x contained in a maximal facet of the apartment of T, then $(\mathrm{Iw}, \mathrm{triv})$ is a type for $\mathrm{Rep}(G)_{[T,\mathrm{triv}]}$, where triv denotes the one-dimensional trivial representation. The group Iw is called an *Iwahori* subgroup, and the corresponding Bernstein block $\mathrm{Rep}(G)_{[T,\mathrm{triv}]}$ is called the *principal block*. The principal block contains the trivial representation of $G(F)$.

4.3 Hecke algebras

A reason for the importance of types is that it leads to explicit modules, called Hecke algebras, such that the Bernstein blocks are equivalent to modules over these algebras, see Theorem 4.3.3. To introduce Hecke algebras, let K be a compact, open subgroup of $G(F)$, and let (ρ, W) be an irreducible smooth representation of K.

Definition 4.3.1. The *Hecke algebra* $\mathcal{H}(G, K, \rho)$ is
the $\mathbb{C}$-vector space of functions $f : G(F) \to \mathrm{End}_{\mathbb{C}}(W)$ satisfying

(a) $f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2)$ for all $k_1, k_2 \in K, g \in G(F)$, and

(b) the support of f is compact

together with the multiplication given by the *convolution* defined by

$$(f_1 * f_2)(g) = \sum_{x \in G(F)/K} f_1(x) f_2(x^{-1}g)$$

for all $f_1, f_2 \in \mathcal{H}(G, K, \rho)$ and $g \in G(F)$.

Here $\mathrm{End}_{\mathbb{C}}(W)$ denotes the $\mathbb{C}$-linear endomorphisms of the $\mathbb{C}$-vector space W, i.e., the endomorphisms are not required to preserve the action of K. Note that

$$\sum_{x \in G(F)/K} f_1(x) f_2(x^{-1}g) = \int_{G(F)} f_1(x) f_2(x^{-1}g) dx$$

if we choose the measure dx to be the Haar measure that satisfies $\int_K 1 dx = 1$.

Exercise 4.3.2. Show that $\mathcal{H}(G, K, \rho) \simeq \mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)} \rho\right)$ where the product structure on the latter is given by composition.

Theorem 4.3.3 ([BK98]). *If (K, ρ) is an $[M, \sigma]$-type, then the Bernstein block $\mathrm{Rep}(G)_{[M,\sigma]}$ is equivalent to the category of right unital $\mathcal{H}(G, K, \rho)$-modules, i.e.,*

$$\mathrm{Rep}(G)_{[M,\sigma]} \simeq Mod - \mathcal{H}(G, K, \rho).$$

The equivalence in the above theorem is given by sending $(\pi, V) \in \mathrm{Rep}(G)_{[M,\sigma]}$ to the nontrivial vector space $\mathrm{Hom}_K(W, V)$. The action of $\mathcal{H}(G, K, \rho) \simeq \mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)} \rho\right)$ on this space is given by using Frobenius reciprocity to identify

$$\mathrm{Hom}_K(W, V) \simeq \mathrm{Hom}_{G(F)}\left(\text{c-ind}_K^{G(F)} W, V\right),$$

and $\mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)} \rho\right)$ acts on the right hand side via precomposition.
Of course this result is only of use if we

(a) know that types exist for the Bernstein blocks that we want to study, and

(b) understand the structure of the resulting Hecke algebras.

The first concern will be answered in Sections 4.5 and 4.6, and the second question will be discussed in Sections 4.4 and 4.7 below.

Before given an explicit example of such Hecke algebras, let us mention one a first fundamental observation about those Hecke algebera: Their support. By the *support* of $\mathcal{H}(G, K, \rho)$, denoted by $\mathrm{Supp}\big(\mathcal{H}(G, K, \rho)\big)$, we refer to all those $g \in G(F)$ such that there exists an element $f \in \mathcal{H}(G, K, \rho)$ with $f(g) \neq 0$. The support is tightly linked with the notion of intertwining, see Notation 3.2.2, as the following exercise shows.

Exercise 4.3.4. $\mathrm{Supp}\big(\mathcal{H}(G, K, \rho)\big) = \{g \in G(F) \,|\, g \text{ intertwines } (K, \rho)\}$

4.4 The Iwahori–Hecke algebra

to be written

4.5 Depth-zero types

In order to use the theory of types to study Bernstein blocks, we need to know that types exist, and ideally we like to have an explicit construction for them. We start by treating the special case of Bernstein blocks that consist of depth-zero representations, generalizing the construction of depth-zero supercuspidal representations.

Let $x \in \widetilde{\mathscr{B}}(G, F)$. Then Moy and Prasad ([MP96, §6.3]) construct a Levi subgroup $M \subseteq G$ with the properties that $x \in \widetilde{\mathscr{B}}(M, F) \subseteq \widetilde{\mathscr{B}}(G, F)$, that x is contained in a facet of minimal dimension of $\widetilde{\mathscr{B}}(M, F)$, and that the inclusion $M_{x,0} \hookrightarrow G_{x,0}$ induces an isomorphism $M_{x,0}/M_{x,0+} \xrightarrow{\simeq} G_{x,0}/G_{x,0+}$. Hence we also have

$$(G_{x,0} \cdot M_x)/G_{x,0+} \simeq M_x/M_{x,0+}.$$

Proposition 4.5.1 ([MP96])**.** *Let ρ^0 be an irreducible representation of $G_{x,0}M_x$ that is trivial on $G_{x,0+}$ and such that $\rho^0|_{G_{x,0}}$ is a cuspidal representation of the reductive group $G_{x,0}/G_{x,0+} \simeq M_{x,0}/M_{x,0+}$. Then the pair $(G_{x,0}M_x, \rho^0)$ is a G-cover of $(M_x, \rho^0|_{M_x})$.*

Proof. This follows from [MP96, Prop 6.7], see also [KY17, 3.3 Prop]. □

In the setting of the previous proposition, let $\widetilde{\rho}^0$ be an irreducible representation of $M_{[x]}$ that is trivial on $M_{x,0+}$ and whose restriction to M_x contains the restriction $\rho^0|_{M_x}$ of ρ^0. Then, by Theorem 3.3.1 and Remark 3.3.2, the representation $\text{c-ind}_{M_{[x]}}^{M(F)} \widetilde{\rho}^0$ is a supercuspidal irreducible representation, and by Fact 4.2.2 the pair $(M_x, \rho^0|_{M_x})$ is an $[M, \text{c-ind}_{M_{[x]}}^{M(F)} \widetilde{\rho}^0]_M$-type. Combining this observation with Proposition 4.5.1 and Fact 4.2.4, we obtain the following corollary.

Corollary 4.5.2. *The pair* $(G_{x,0}M_x, \rho^0)$ *is an* $\left[M, \text{c-ind}_{M_{[x]}}^{M(F)} \widetilde{\rho}^0\right]_G$*-type.*

Note that all irreducible representations in $\mathrm{Rep}(G)_{\left[M, \text{c-ind}_{M_{[x]}}^{M(F)} \widetilde{\rho}^0\right]}$ have depth zero, so we may refer to such a Bernstein block also as a *depth-zero Bernstein block*, and to the pair $(G_{x,0}M_x, \rho^0)$ as a *depth-zero type*.

4.6 Types constructed by Kim and Yu – with a twist

Based on the construction of depth-zero types, we will now construct types for Bernstein blocks consisting of representations of positive depth, based on the work of Kim and Yu ([KY17]), but twisted by a quadratic character arising from the work of Fintzen, Kaletha and Spice ([FKS23]).

The construction of types by Kim and Yu takes as an input a datum very similar to the supercuspidal datum introduced in Section 3.7 except that we weaken some of the assumptions. In order to do so, we first explain how to construct from a a sequence $G_1 = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_{n+1}$ of twisted Levi subgroups of G with G_{n+1} split over a tamely ramified extension and a point x in the building of $\widetilde{\mathscr{B}}(G_{n+1}, F) \subset \widetilde{\mathscr{B}}(G, F)$ a Levi subgroup M_i for each G_i ($1 \leq i \leq n+1$). We let $M_{n+1} \subseteq G_{n+1}$ be the Levi subgroup as introduced by Moy and Prasad in [MP96, §6.3], see Section 4.5, that satisfies that $(M_{n+1})_{x,0}/(M_{n+1})_{x,0+} \xrightarrow{\simeq} (G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ and that x is a facet of minimal dimension in $\widetilde{\mathscr{B}}(M_{n+1}, F)$. We let $Z_{\text{split}}(M_{n+1})$ be the maximal split torus in the center of M_{n+1} and let $M_i := \mathrm{Cent}_{G_i}(Z_{\text{split}}(M_{n+1}))$ be its centralizer in G_i. We also write $M := M_1$.

Now we can modify the input for the construction of supercuspidal representations to an input for the construction of types as follows. Let $((G_i)_{1 \leq i \leq n+1}, x, (r_i)_{1 \leq i \leq n}, \rho^0, (\phi_i)_{1 \leq i \leq n})$ for some non-negative integer n be the following data

(a) $G = G_1 \supseteq G_2 \supsetneq G_3 \supsetneq \ldots \supsetneq G_n \supsetneq G_{n+1}$ are twisted Levi subgroups of G that split over a tamely ramified extension of F,

(b) $x \in \widetilde{\mathscr{B}}(M_{n+1}, F) \subset \widetilde{\mathscr{B}}(G_{n+1}, F) \subset \widetilde{\mathscr{B}}(G, F)$,

(c) $r_{n-1} > r_2 > \ldots > r_n > 0$ are real numbers,

(d) ϕ_i, for $1 \leq i \leq n$, is a character of $G_{i+1}(F)$ of depth r_i,

(e) ρ^0 is an irreducible representation of $(G_{n+1})_{x,0} \cdot (M_{n+1})_x$ that is trivial on $(G_{n+1})_{x,0+}$,

satisfying the following conditions

(i) (no condition on $Z(G_{n+1})/Z(G)$)

(ii) the point x is chosen so that $(M_i)_{x,r_i/2}/(M_i)_{x,r_i/2+} \xrightarrow{\simeq} (G_i)_{x,r_i/2}/(G_i)_{x,r_i/2+}$ for $1 \leq i \leq n$, where M_i is defined as above,

(iii) ϕ_i is (G_i, G_{i+1})-generic (relative to x) of depth r_i for all $1 \leq i \leq n$,

(iv) $\rho^0|_{(G_{n+1})_{x,0}}$ is a cuspidal representation of the reductive group $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+} \simeq (M_{n+1})_{x,0}/(M_{n+1})_{x,0+}$.

We will call a tuple satisfying the above conditions a *G-datum*. We also set

$$G^0 := G_{n+1}, \quad M^0 := M_{n+1}, \quad \text{and } K^0 := G^0_{x,0} \cdot M^0_x.$$

Then by the previous subsection, Section 4.5, the pair (K^0, ρ^0) is an $[M^0, \sigma^0]_G$-type for some depth-zero supercuspidal representation σ^0 of $M^0(F)$.

Note that in these lecture notes we assume by definition that $K^0 = G^0_{x,0} \cdot M^0_x$ while [AFMOb] allows for a more general choice for K^0 that is recorded as part of the G-datum in [AFMOb, Definition 4.1.1 and (4.1.2)].

Remark 4.6.1. We use the same conventions here as in Section 3.7 to allow for an easier comparison. The reference [AFMOb] uses the convention of Yu ([Yu01]) and Kim and Yu ([KY17]). Here is how to obtain our notation used here from the one used in [AFMOb]:

$$\begin{aligned}
(G_{n+1}, G_n, \ldots, G_2, G_1 = G) &= \begin{cases} (G^0, G^1, \ldots, G^d = G) & \text{if } r_{d-1} = r_d \\ (G^0, G^1, \ldots, G^d, G^d = G) & \text{if } r_{d-1} \neq r_d \end{cases} \\
(r_n, r_{n-1}, \ldots, r_2, r_1) &= \begin{cases} (r_0, r_1, \ldots, r_{d-1}) & \text{if } r_{d-1} = r_d \\ (r_0, r_1, \ldots, r_{d-1}, r_d) & \text{if } r_{d-1} \neq r_d \end{cases} \\
(\phi_n, \phi_{n-1}, \ldots, \phi_2, \phi_1) &= \begin{cases} (\phi_0, \phi_1, \ldots, \phi_{d-1}) & \text{if } r_{d-1} = r_d \\ (\phi_0, \phi_1, \ldots, \phi_{d-1}, \phi_d) & \text{if } r_{d-1} \neq r_d\,, \end{cases}
\end{aligned}$$

where the condition $r_{d-1} = r_d$ or $r_{d-1} \neq r_d$ refers to the rational numbers r_{d-1} and r_d in [AFMOb] and we write $r_{-1} = 0$.

While a case distinction is necessary to translate between the two conventions, the convention used here has the advantage that we need to make less case distinctions in the actual definition of a G-datum (c.f., [AFMOb, Definition 4.1.1 D5]). Moreover, we also do not need a case distinction when extracting the Heisenberg–Weil datum from a G-datum that is used to construct types, see [AFMOb, Remark 4.1.4] and the formulas preceding it. Using our convention, the Heisenberg–Weill datum attached to a G-datum is simply the G-datum without ρ^0.

For constructions of types rather than obtaining an exhaustion result, where our convention of notation was introduced ([Fin21d]), replacing all our indices i by $n+1-i$ might be desirable, but we were worried that introducing yet a third convention would cause more confusion than benefits. However, we do use the notation $G^0 = G_{n+1}$ for the group that will play an important role in the reduction to depth-zero results in Section 4.7.

Let $((G_i)_{1 \leq i \leq n+1}, x, (r_i)_{1 \leq i \leq n}, \rho^0, (\phi_i)_{1 \leq i \leq n})$ be a G-datum. To this G-datum we attach the pair (K, ρ) with

$$K = (G_1)_{x,\frac{r_1}{2}} (G_2)_{x,\frac{r_2}{2}} \ldots (G_n)_{x,\frac{r_n}{2}} \cdot K^0 \quad \text{and} \quad \rho = \rho^0 \otimes \kappa^{\mathrm{nt}} \otimes \epsilon = \rho^0 \otimes \kappa,$$

where ρ^0 also denotes the extension of ρ^0 to K that is trivial on $(G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}}\dots(G_n)_{x,\frac{r_n}{2}}$, the representation κ^{nt} is constructed from the characters $\phi_1,\dots,\phi_n$ via the theory of Heisenberg–Weil representations analogous to the construction in Section 3.8, the representation ϵ is a quadratic character arising from [FKS23], see also Section 3.10, and $\kappa := \epsilon\kappa^{\mathrm{nt}}$. See [AFMOb, §4.1] for more details on the construction of κ.

Theorem 4.6.2 ([KY17] (and [Fin21a] or [FKS23])). *The pair (K,ρ) is an $[M,\sigma]$-type (for some supercuspidal representation σ of $M(F)$).*

The theorem is proven using the theory of covers discussed in Section 4.2. Under minor tameness assumptions this construction provides us with types for every Bernstein block. More precisely, we have the following result.

Theorem 4.6.3 ([KY17, Fin21d]). *Suppose that G splits over a tamely ramified field extension of F and that p does not divide the order of the absolute Weyl group of G. Then for every $[M,\sigma]\in\mathfrak{I}(G)$, there exists a G-datum whose associated pair (K,ρ) by the construction above is an $[M,\sigma]$-type.*

This theorem is proven in the same way as Theorem 3.11.1, the result that under the same assumptions all supercuspidal representations arise from Yu's construction. In fact, Theorem 3.11.1 is proven by first proving Theorem 4.6.3.

4.7 Structure of Hecke algebras

From now on let $((G_i)_{1\le i\le n+1}, x, (r_i)_{1\le i\le n}, \rho^0, (\phi_i)_{1\le i\le n})$ be a G-datum and let (K,ρ) be the corresponding $[M,\sigma]$-type from Section 4.6. Recall that we write $G^0 = G_{n+1}$ and that (K^0,ρ^0) is an $[M^0,\sigma^0]_G$-type. In this section we will discuss an isomorphism between $\mathcal{H}(G,K,\rho)$ and $\mathcal{H}(G,K^0,\rho^0)$ and use it to describe the structure of $\mathcal{H}(G,K,\rho)$, following [AFMOa, AFMOb].

In order to understand the Hecke algebras, we first introduce some more notation in line with [AFMOb]. We set

$$\begin{aligned} K_{M^0} &:= K\cap M^0(F) = K^0\cap M^0(F),\\ \rho_{M^0} &:= \rho^0|_{K_{M^0}},\\ N(\rho_{M^0})_{[x]_{M^0}} &:= \{n\in G^0(F)\,|\, nM^0n^{-1}=M^0,\, nK^0_Mn^{-1}=K^0_M,\, {}^n\rho_{M^0}\simeq\rho_{M^0}\}. \end{aligned}$$

Our isomorphism between the two Hecke algebras $\mathcal{H}(G,K,\rho)$ and $\mathcal{H}(G,K^0,\rho^0)$ will be constructed in a support-preserving way. To make sense of such a statement, we first observe the following structure of their supports.

Fact 4.7.1. *We have* $\mathrm{Supp}\left(\mathcal{H}(G,K,\rho)\right) = K\cdot\mathrm{Supp}\left(\mathcal{H}(G^0,K^0,\rho^0)\right)\cdot K\subset K\cdot N(\rho_{M^0})_{[x]_{M^0}}\cdot K$.

Fact 4.7.2. *Let $g\in\mathrm{Supp}(\mathcal{H}(G,K,\rho))$. The $\mathbb{C}$-subspace of functions in $\mathcal{H}(G,K,\rho))$ that are supported on Kg has dimension one.*

While the support is a priori only a set of double cosets, the next result allows us to endow it with a group structure.

Proposition 4.7.3 ([AFMOb]). *There exists a subgroup $N^\heartsuit$ of $N(\rho_{M^0})_{[x]_{M^0}}$ that is contained in the support of $\mathcal{H}(G,K,\rho)$ such that the inclusion map induces a bijection*

$$N^\heartsuit/(N^\heartsuit \cap M_x^0) \xrightarrow{\simeq} K\backslash \operatorname{Supp}\big(\mathcal{H}(G,K,\rho)\big)/K.$$

Since the intersection $(N^\heartsuit \cap M_x^0)$ is a normal subgroup of $N^\heartsuit$, the quotient $N^\heartsuit/(N^\heartsuit \cap M_x^0)$ is a group, which we denote by $W^\heartsuit$. This equips the support of our Hecke algebras with a group structure and allows us to describe the Hecke algebra structure as follows.

Theorem 4.7.4 ([AFMOb]). *The group $W^\heartsuit$ admits the structure of a semi-direct product $W^\heartsuit \simeq \Omega(\rho_{M^0}) \ltimes W(\rho_{M^0})_{\mathrm{aff}}$ where $W(\rho_{M^0})_{\mathrm{aff}}$ is an affine Weyl group and such that*

$$\mathcal{H}(G,K,\rho) \simeq \mathbb{C}[\Omega(\rho_{M^0}),\mu] \ltimes \mathcal{H}_{\mathrm{aff}}(W(\rho_{M^0})_{\mathrm{aff}}, q_s)$$

for some 2-cocycle $\mu : \Omega(\rho_{M^0}) \times \Omega(\rho_{M^0}) \to \mathbb{C}$ and some $q_s \in \mathbb{Q}_{>1}$ with the index s running through a set of simple reflection of the affine Weyl group $W(\rho_{M^0})_{\mathrm{aff}}$. Here $\mathbb{C}[\Omega(\rho_{M^0}),\mu]$ denotes the twisted group algebra, i.e., $\mathbb{C}[\Omega(\rho_{M^0}),\mu] \simeq \bigoplus_{t\in\Omega(\rho_{M^0})} \mathbb{C}b_t$ as a vector space with multiplication given by $b_{t_1}b_{t_2} = \mu(t_1,t_2)b_{t_1t_2}$.

[[section to be expanded and more explanation added, need to define $\mathcal{H}_{\mathrm{aff}}(W(\rho_{M^0})_{\mathrm{aff}}, q_s)$]]

Theorem 4.7.5 ([AFMOb]). *There exists a representation $\widetilde{\kappa} : N^\heartsuit \cdot (K \cap M(F)) \to \operatorname{End}(V_\kappa)$ such that $\widetilde{\kappa}|_{K\cap M(F)} = \kappa|_{K\cap M(F)}$ and such that there exists an algebra-isomorphism*

$$\mathcal{I} : \mathcal{H}(G^0,K^0,\rho^0) \xrightarrow{\simeq} \mathcal{H}(G,K,\rho)$$

defined by the following:

If $\varphi \in \mathcal{H}(G^0,K^0,\rho^0)$ is supported on K^0nK^0 with $n \in N^\heartsuit$, then $\mathcal{I}(\varphi)$ is supported on KnK and

$$\mathcal{I}(\varphi)(n) = d_n \cdot \varphi(n) \otimes \widetilde{\kappa}(n) \quad \text{with} \quad d_n = \sqrt{\frac{|K^0/(nK^0n^{-1}\cap K^0)|}{|K/(nKn^{-1}\cap K)|}}.$$

Selected notation

Selected terminology

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