Reduction from representations of p-adic groups to representations of finite groups

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Abstract. This article sketches how the category of smooth representations of p-adic groups decomposes into blocks and how each block is equivalent to a block of a (smaller) p-adic group that consists entirely of depth-zero representations, i.e., those representations of a p-adic group that roughly correspond to representations of finite groups of Lie type. This allows to reduce problems about arbitrary p-adic groups, including problems in the Langlands program, to the much better understood and more well studied representations of depth zero.

This article surveys two very different approaches to achieve these results based on what coefficients are used for the representations. The first one (based on two preprints of the author with Jeffrey Adler, Manish Mishra and Kazuma Ohara from 08/2024) treats the case of representations with complex coefficients using the Bernstein decomposition and type theory, obtaining the desired equivalences of blocks via providing explicit isomorphisms of Hecke algebras. The second is joint work in progress with Jean-François Dat and treats the case of representations with R-coefficients for any ring R that contains all p-power roots of unity, a fourth root of unity, and the inverse of a square-root of p. This includes, for example, the case where R is the ring $\bar{\mathbb{Z}}[1/p]$ or where R is an algebraically closed field of characteristic different from p. In this case we decompose the category into a product of subcategories via idempotents constructed from wild inertia parameters and we prove the equivalence between arbitrary blocks and depth-zero blocks using appropriate equivariant coefficient systems on the Bruhat–Tits building.

All results are obtained under a tameness assumption.

1 Introduction. Representations of p-adic groups, e.g., of $\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{SL}_n(\mathbb{Q}_p), \mathrm{SO}_n(\mathbb{Q}_p)$ or $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$, are an intensely studied area of mathematics that also plays a key role in the Langlands program and has applications to automorphic forms (generalizations of modular forms), among others. This article surveys the current state of the art of our understanding of the structure of the whole category of representations of p-adic groups, including recent and forthcoming results that relate the blocks of this category to depth-zero blocks of another p-adic group. Depth-zero representations, introduced in Section 2 below, roughly correspond to representations of finite groups of Lie type and are much better understood than general representations of p-adic groups. Relating arbitrary blocks to depth-zero blocks therefore allows us to reduce many problems about representations of p-adic groups and the Langlands program to depth-zero representations, where the answer is either already known or easier to achieve. In this survey we study both, complex representations of p-adic groups and representations of p-adic groups with much more general coefficients, e.g., representations on $\bar{\mathbb{Z}}[1/p]$-modules or on vector spaces over an algebraically closed field of positive characteristic different from ℓ. The techniques used in both cases are quite different, but yet rely on some similar structural results.

Complex representations. We know for more than 50 years that complex representations of p-adic groups[1] are made out of certain buildings blocks that are called *supercuspidal representations*. In 1984, Bernstein [Ber84] showed that appropriate equivalence classes of these supercuspidal representations of subgroups of our p-adic group G also serve to decompose the whole category of the representations of G into blocks, i.e., into indecomposable

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The author was partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement n° 950326).
Mathematics Subject Classification 2020: 22E50; 11F27, 22E35, 20C08, 20C20
last modified: October 4, 2025; the most up to date version can be accessed at https://www.math.uni-bonn.de/people/fintzen/research
[1]For the expert, we only consider smooth representations and by a p-adic group we mean the F-points of a reductive group over a non-archimedean local field F. For a reader less experienced with this area, we explain these details in the main part of the paper.

subcategories, called *Bernstein blocks*, whose product is the whole category:

$$\underbrace{\mathrm{Rep}_{\mathbb{C}}(G)}_{\text{complex representations of } G} = \prod_{(M,\sigma)/\sim} \underbrace{\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}}_{\text{Bernstein block}}. \tag{1.1}$$

Here σ is a supercuspidal representation of a subgroup M of G. We refer the reader to Section 3.1 for a more detailed explanation of the notation.

Thus the Bernstein decomposition reduced the task of describing the whole category of representations of p-adic groups into two sub-tasks:

PROBLEM 1.1. *Construct all supercuspidal representations.*

PROBLEM 1.2. *Describe the Bernstein block* $\mathrm{Rep}(G)_{[M,\sigma]}$ *explicitly.*

Problem 1.1 has been solved under a tameness assumption thanks to the work of a lot of mathematicians over the past 50 years. A brief overview of the answer aimed at a general mathematics audience is contained in the recent survey article [Fin25] and a more detailed account is given in [Fin23]. A reader already familiar with the notion of p-adic groups and smooth representations can refer to [Fin] for a more detailed survey of the answer to Problem 1.1 including the construction of supercuspidal representations with some sketches of proofs. For a survey on how they fit into the local Langlands correspondence we refer the reader to [Kal23].

Given that the current state of the art regarding Problem 1.1 is covered in the above surveys, the goal of this article is to complement these surveys by answering Problem 1.2. This is the subject of Section 3, of which we provide a brief overview here.

In 1998, Bushnell and Kutzko [BK98] introduced a technique to study Bernstein blocks that is known as *type theory*. It requires to first prove the existence of *types* (see Section 3.2 for their definition) for each Bernstein block, which under a tameness assumption is known, see Theorem 3.11. We therefore assume from now on for the remainder of the introduction that the tameness assumption is satisfied, which, for the curious expert, means that the group G splits over a tamely ramified field extension and that p does not divide the order of the absolute Weyl group of G.

From the existence of types, one deduces that the Bernstein blocks are equivalent to modules over an algebra constructed from the types, and it remains to understand these algebras and the category of modules over them. Two recent joint preprints of the author with Jeffrey Adler, Manish Mishra and Kazuma Ohara [AFMOa, AFMOb] that we discuss in Section 3.7, see also Section 3.8 for the discussion of some prior work, provide an explicit description of these algebras in terms of generators and relation. More precisely, these algebras turn out to be semi-direct products of twisted group algebras with affine Hecke algebras:

$$\mathcal{H}(G, K, \rho) \simeq \mathbb{C}[\Omega(\rho_{M^0}), \mu] \ltimes \mathcal{H}_{\mathrm{aff}}(W(\rho_{M^0})_{\mathrm{aff}}, \{q_s\}).$$

We refer the reader to Section 3.7 for an explanation of the notation. Since modules over affine Hecke algebra are well-studied and the same applies to modules over twisted group algebras, this provides us with a very explicit understanding of the Bernstein blocks.

Moreover, by proving an isomorphism of appropriate algebras attached to types, we obtain an equivalence between arbitrary Bernstein blocks and depth-zero Bernstein blocks, see Theorems 3.16 and 3.18 for details.

Since the reduction-to-depth-zero is obtained via an explicit isomorphism of explicitly described algebras attached to explicit types, and since the equivalence between Bernstein blocks and modules over these algebras is also very explicit, we expect the result to be quite powerful to reduce also explicit problems about representations of p-adic groups to depth-zero, e.g., the question if a given parabolically induced representation is irreducible or not.

Representations with more general coefficients. Motivated by applications to automorphic forms and applications to number theory, mathematicians started studying representations with coefficients in algebraically closed fields of characteristic ℓ for some prime number ℓ. See Vigneras's ICM proceedings article from 2002 [Vig02]. Nowadays these representations also feature in more geometric constructions of the Langlands correspondence. Since the case of $\ell = p$ behaves very differently from the case $\ell \neq p$ and from complex representations, we focus on the case $\ell \neq p$ in this article. From now on assume that $\mathcal{C}$ is an algebraically closed field of characteristic $\ell \neq p$. Similarly to the complex setting above, the $\mathcal{C}$-representations are made out of building blocks, called *cuspidal* representations, and we know how to construct all of them under a tameness assumption [Fin22], i.e., we can answer Problem 1.1 above also for $\mathcal{C}$-representations. However, while an analogue of the Bernstein decomposition (1.1) holds in special cases, e.g., for $G = \mathrm{GL}_n$ by Vigneras [Vig98], or if the prime ℓ is sufficiently large compared to p by Dat–Helm–Kurinczuk–Moss [DHKM24, Theorem 4.22], the Bernstein decomposition does not work for general $\mathcal{C}$-representations of general p-adic groups. This means the equivalence classes of pairs of subgroups and cuspidal representations (M, σ) do not divide the category of all $\mathcal{C}$-representation into a product of subcategories. There might be non-trivial extensions between representations arising from different, non-equivalent pairs (M, σ). Relatedly, the above approach via type theory and Hecke algebras does not work in general for $\mathcal{C}$-representations.

In the last ten years, studying representations over more general commutative rings R has received more attention, also because of their presence in geometric constructions of the Langlands correspondence [FS21]. Working over rings R like $\bar{\mathbb{Z}}[1/p]$ allows to "interpolate" between $\bar{\mathbb{F}}_\ell$-representations for varying prime numbers ℓ and thereby combines features observed when working mod ℓ for different primes ℓ. These $\bar{\mathbb{Z}}[1/p]$-representations also appear when trying to construct a local Langlands correspondence in families [EH14, HM18, DHKM24]. For a survey of the state of the art of representations of p-adic groups with coefficients in a commutative ring R of a few years ago see Vigneras's 2022 ICM Noether Lecture survey article [Vig23].

However, at the time of Vigneras's article, it was still unclear how to decompose the category $\mathrm{Rep}_R(G)$ of all R-representations of a general p-adic group G into blocks and how to show that every block is equivalent to a depth-zero block. Last year, Helm–Kurinczuk–Skodlerack–Stevens [HKSS24] obtained a block decomposition for $\bar{\mathbb{Z}}[1/p]$-representations of p-adic classical groups (i.e., symplectic, orthogonal and unitary groups) and their inner forms assuming $p \neq 2$, using their relation to GL_n. In recent joint work of the author and Jean-François Dat [DF], we use a different approach to obtain a block decomposition for all p-adic groups, including those of exceptional type, under the tameness assumption mentioned above, and, in addition, we prove that arbitrary blocks are equivalent to depth-zero blocks. Section 4 provides a survey of these results. This will then allow to reduce many questions about R-representations, and related questions in the Langlands program, to depth-zero representations.

The rings R that we consider in [DF] are those in which p is invertible and which contain all p-power roots of unity. The assumption that p is invertible implies that we allow the case $R = \bar{\mathbb{F}}_\ell$ for $\ell \neq p$, but exclude the case of $\bar{\mathbb{F}}_p$, as mentioned above. The assumption about containing p-power roots of unity arises from our p-adic group containing large pro-p-groups, whose characters are valued in the p-power roots of unity. For technical reason, we also ask R to contain a fourth root of unity and a square-root of p. To give a few examples, R could be the field $\bar{\mathbb{F}}_\ell$, the field $\mathbb{C}$, or the ring $\bar{\mathbb{Z}}[1/p]$.

Our approach to decompose the category $\mathrm{Rep}_R(G)$ into subcategories that are indecomposable if $R = \bar{\mathbb{Z}}[1/p]$ is motivated by the expectation that the depth-zero representations should correspond to Langlands parameter (which are roughly representations of the absolute Galois group of the local field we consider, e.g., of the p-adic numbers) whose restriction to the wild inertia subgroup are trivial. Following this expectation, we parameterize our subcategories $\mathrm{Rep}_R(G)^{[\phi,\iota]}$ by wild inertia parameters ϕ, see Section 4.1 for the precise definition, and some auxiliary data denoted by ι, which we define in Section 4.2. For the experts, the ι distinguishes between subgroups of G whose depth-zero blocks are our target and which are pure inner forms of each other, i.e., they have the same Langlands dual group. Parameterizing subcategories of representations of p-adic groups by restrictions of Langlands parameters had already been advertised by Dat about ten years earlier in [Dat17], where he demonstrated the idea by reparametrizing the Bernstein blocks for GL_n in terms of restrictions of Langlands parameters to the inertia subgroup and pointed out that the Langlands parameters that are trivial on wild inertia correspond to depth-zero (complex, $\bar{\mathbb{F}}_\ell$-, or $\bar{\mathbb{Z}}_\ell$-) representation of GL_n.

Using the above indicated parameterization, we obtain the following decomposition of the category of R-representations of a general tame p-adic group G, see Theorems 4.1 and 4.2 for details,

$$\mathrm{Rep}_R(G) \simeq \prod_{\{(\phi,\iota)\}/\sim} \mathrm{Rep}_R(G)^{[\phi,\iota]}, \tag{1.2}$$

and for each indexing pair (ϕ, ι) there exists a subgroup $G_\iota \subseteq G$ such that we have an equivalence of categories

$$\mathrm{Rep}_R(G)^{[\phi,\iota]} \simeq \mathrm{Rep}_R(G_\iota)_{\text{depth-zero}}, \tag{1.3}$$

where $\mathrm{Rep}_R(G_\iota)_{\text{depth-zero}}$ denotes the category of depth-zero R-representation of the p-adic group G_ι. Since Dat and Lanard [DL25] have recently shown that all the depth-zero $\bar{\mathbb{Z}}[1/p]$-representations form a single block, i.e., this subcategory cannot be written as a product of two non-trivial subcategories, the decomposition in (1.2) is indeed a block decomposition for $R = \bar{\mathbb{Z}}[1/p]$. The subcategories $\mathrm{Rep}_R(G)^{[\phi,\iota]}$ are constructed using appropriate idempotents constructed from (ϕ, ι) and indexed by the Bruhat–Tits building, see Section 4.3 for more details. The equivalence-to-depth-zero (1.3) is obtained via the theory of equivariant coefficient systems on the Bruhat–Tits building and involves the construction of an auxiliary coefficient system that induces the equivalence between the positive-depth coefficient system on the building for G and the depth-zero coefficient system on the building for G_ι, analogous to how an appropriate bi-module induces an equivalence between the categories of modules over two different rings. The auxiliary coefficient system relies on the theory of Heisenberg–Weil representations over R and crucially uses the existence of a quadratic character as introduced in [FKS23] to obtain the needed compatibilities between different Heisenberg–Weil representations. The notion of coefficient systems is introduced in Section 4.5 and a few more details about the proof of (1.3) are sketched in Section 4.6.

Updated version of this paper. Unfortunately the joint work of the author with Jean-François Dat [DF] was not yet publicly available by the time when this ICM proceedings article needed to be submitted. This might potentially lead to some of the references to precise statement numbers in [DF] changing during the finalization of [DF]. The author will update the present article on her homepage as soon as [DF] is posted publicly. Hence, please see https://www.math.uni-bonn.de/people/fintzen/research.html for the most up-to-date version of the present article. Comments and corrections are more than welcome. Please send them to fintzen@math.uni-bonn.de.

Structure of the paper. In Section 2 we introduce depth-zero representations. In Section 3 we study the block decomposition and reduction-to-depth-zero for complex representations, and in Section 4 we study the same question for representations with more general rings as coefficients, which requires very different techniques, as outlined in the introduction above.

Notation. Throughout the article, p denotes a prime number, $\mathbb{F}_p$ denotes the finite field with p elements, $\mathbb{Z}_p$ denotes the p-adic integers and $\mathbb{Q}_p$ denotes the p-adic numbers. For a a brief introduction to what the p-adic integers and numbers are see [Fin25, §1], for example.

2 Depth-zero representations.

This section introduces depth-zero representations, which are those representations of p-adic groups that roughly correspond to representations of finite groups of Lie type. We start with an explicit example.

2.1 An example of a depth-zero representation.

Let $\mathcal{C}$ be an algebraically closed field of characteristic $\ell \neq p$. Consider the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ of two-by-two matrices of determinant 1 with entries in the finite field $\mathbb{F}_p$. Let ρ be an *irreducible $\mathcal{C}$-representation* of $\mathrm{SL}_2(\mathbb{F}_p)$, i.e., a group homomorphism $\rho : \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{Aut}_{\mathcal{C}}(V_\rho)$, where $\mathrm{Aut}_{\mathcal{C}}(V_\rho)$ denotes the $\mathcal{C}$-linear automorphisms of a finite-dimensional vector space V_ρ over $\mathcal{C}$, with the property that exactly two subspaces ($\{0\}$ and V_ρ itself) of V_ρ are preserved under $\rho(g)$ for all $g \in \mathrm{SL}_2(\mathbb{F}_p)$. We also require ρ to be *cuspidal*, which means that it is not a subrepresentation of a proper parabolic induction, i.e., explicitly, this means that ρ is not a subrepresentation of $\mathrm{Ind}^{\mathrm{SL}_2(\mathbb{F}_p)}_{g\left\{\left(\begin{smallmatrix} t & x \\ 0 & t^{-1}\end{smallmatrix}\right) \mid x \in \mathbb{F}_p, t \in \mathbb{F}_p^\times\right\}g^{-1}} \chi(t)$ for any $g \in \mathrm{SL}_2(\mathbb{F}_p)$ and any group homomorphism $\chi : \mathbb{F}_p^\times \to \mathcal{C}^\times$, or, equivalently, that for every $v \in V_\rho \setminus \{0\}$ we have $\rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)(v) \neq v$. These representations have dimension $p-1$ or $(p-1)/2$, and, if $p > 3$, then these are exactly the irreducible representations of dimension $p-1$ and $(p-1)/2$.

From this representation ρ of the finite group of Lie type $\mathrm{SL}_2(\mathbb{F}_p)$ we can construct a representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ as follows. We also denote by ρ the following composition of group homomorphisms

$$\mathrm{SL}_2(\mathbb{Z}_p) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}_p)/\begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}_{\det=1} \simeq \mathrm{SL}_2(\mathbb{F}_p) \xrightarrow{\rho} \mathrm{Aut}_{\mathcal{C}}(V_\rho).$$

Consider the following infinite dimensional $\mathcal{C}$-vector space

$$V_{\pi_\rho} := \text{c-ind}_{\mathrm{SL}_2(\mathbb{Z}_p)}^{\mathrm{SL}_2(\mathbb{Q}_p)} V_\rho := \left\{ f : \mathrm{SL}_2(\mathbb{Q}_p) \to V_\rho \;\middle|\; \begin{array}{l} f(kg) = \rho(k)(f(g)) \;\forall g \in \mathrm{SL}_2(\mathbb{Q}_p), k \in \mathrm{SL}_2(\mathbb{Z}_p) \\ f \text{ is compactly supported} \end{array} \right\}$$

together with the group homomorphism $\pi_\rho : \mathrm{SL}_2(\mathbb{Q}_p) \to \mathrm{Aut}_{\mathcal{C}}(V_{\pi_\rho})$ given by

$$(\pi_\rho(g)(f))(x) = f(xg) \;\forall\, x, g \in \mathrm{SL}_2(\mathbb{Q}_p), f \in V_{\pi_\rho}.$$

Then π_ρ is an irreducible representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ that is *smooth*, i.e., every vector in V_{π_ρ} is fixed by an open subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Moreover, π_ρ is *cuspidal*, which means that it is not a subrepresentation of a proper parabolic induction, i.e., not a subrepresentation of $\mathrm{Ind}^{\mathrm{SL}_2(\mathbb{Q}_p)}_{g\left\{\left(\begin{smallmatrix} t & x \\ 0 & t^{-1}\end{smallmatrix}\right) \mid x\in\mathbb{Q}_p, t\in\mathbb{Q}_p^\times\right\}g^{-1}} \chi(t)$ for any $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ and any group homomorphism $\chi : \mathbb{Q}_p^\times \to \mathcal{C}^\times$. In addition, π_ρ is of *depth zero*, which means that it has non-zero fixed vectors under one of the following two groups (in this case under the first group):

$$\begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}_{\det=1} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}_{\det=1}\begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

2.2 Depth-zero representations of simply-connected groups. The above example generalizes to general p-adic groups. From now on we fix a non-archimedean local field F of residual characteristic p, i.e., a finite extension of $\mathbb{Q}_p$ or of the field of Laurent series $\mathbb{F}_p((t))$ over the finite field $\mathbb{F}_p$. For simplicity, let us start with a group G that is simply-connected semisimple, e.g., SL_n or a symplectic group Sp_{2n}, or a simply-connected group of exceptional type like E_8. A reader less familiar with general reductive groups is invited to just think about the examples provided, e.g., the special linear group SL_n and the symplectic group Sp_{2n} in this subsection. A brief introduction to reductive groups can be found in [Fin23, §2].

Definition 2.1 (Depth-zero representation of simply-connected group). *An irreducible smooth representation π of $G(F)$ has depth zero if it has non-zero fixed vectors under the pro-p-radical of a maximal compact subgroup of $G(F)$.*

In the case of $G(F) = \mathrm{SL}_2(\mathbb{Q}_p)$, every maximal compact subgroup of $G(F)$ is a conjugate of either $\mathrm{SL}_2(\mathbb{Z}_p)$ or of $\left(\begin{smallmatrix} 0 & 1 \\ p & 0\end{smallmatrix}\right) \mathrm{SL}_2(\mathbb{Z}_p) \left(\begin{smallmatrix} 0 & p^{-1} \\ 1 & 0\end{smallmatrix}\right)$, and their pro-$p$-radicals are conjugates of the groups in (2.1).

It turns out ([MP94, MP96] for $\mathcal{C} = \mathbb{C}$) that all depth-zero, cuspidal irreducible representations are of the form

$$V_{\pi_\rho} := \text{c-ind}_K^{G(F)} V_\rho := \left\{ f : G(F) \to V_\rho \;\middle|\; \begin{array}{l} f(kg) = \rho(k)(f(g)) \ \forall g \in G(F), k \in K \\ f \text{ is compactly supported} \end{array} \right\}$$

for some maximal compact subgroup $K \subseteq G(F)$ with pro-p-radical K_+ and a representation ρ of the form

$$\rho : K \twoheadrightarrow K/K_+ \xrightarrow{\text{irreducible, cuspidal}} \mathrm{Aut}_{\mathcal{C}}(V_\rho).$$

Note that K/K_+ is a finite group of Lie type, more precisely the $\mathbb{F}_q$-points of a connected reductive group where $\mathbb{F}_q$ denotes the residue field of F. Hence it make sense to talk about cuspidal representations of the group K/K_+, which are those not contained in a proper parabolic induction. In this way, depth-zero representations are tightly linked to representations of finite groups of Lie type.

2.3 Depth-zero representations and the Moy–Prasad filtration. From now on we consider the general setting that G is a connected reductive group, e.g., $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$, or a group of exceptional type, e.g., of type E_8, over the non-archimedean local field F. We will assume throughout this article that G splits over a tamely ramified extension of F.

In the 1990s, Moy and Prasad [MP94, MP96] defined a filtration of $G(F)$ by compact, open subgroups

$$G_x \trianglerighteq G_{x,0} \triangleright G_{x,r_1} \triangleright G_{x,r_2} \triangleright G_{x,r_3} \triangleright \ldots,$$

where x is a point in the (extended) Bruhat–Tits building $\mathcal{B}(G,F)$ of G, a building introduced by Bruhat and Tits [BT72, BT84] that helps to classify maximal, compact subgroups of $G(F)$, and where $0 < r_1 < r_2 < r_3 < \ldots$ are real numbers depending on x. We refer the reader to [Fin, §2] for more details, and to [Fin25, §2.4] for a very brief overview. For any non-negative real number r, we write $G_{x,r} := \cup_{r \le r_i} G_{x,r_i}$ and $G_{x,r+} := \cup_{r<r_i} G_{x,r_i}$. The group $G_{x,0+}$ is the pro-p-radical of $G_{x,0}$ and $G_{x,0}/G_{x,0+}$ are the $\mathbb{F}_q$-points of a connected reductive group. If G is simply-connected semisimple, then $G_x = G_{x,0}$, and the group G_x is a maximal compact subgroup of $G(F)$ if and only if x is a vertex in the Bruhat–Tits building.

Definition 2.2 (Depth-zero irreducible representation). *An irreducible smooth representation π of $G(F)$ has depth zero if it has non-zero fixed vectors under $G_{x,0+}$ for some $x \in \mathcal{B}(G,F)$.*

All depth-zero cuspidal representations of $G(F)$ arise via compact induction from the normalizer $N_{G(F)}(G_{x,0})$ of the subgroup $G_{x,0}$ of a representation whose restriction to $G_{x,0}$ is the inflation of a cuspidal representation of $G_{x,0}/G_{x,0+}$.

Note that the above condition for an irreducible smooth representation $\pi : G(F) \to \mathrm{Aut}_{\mathcal{C}}(V_\pi)$ to be of depth zero is equivalent to the condition that $V_\pi = \sum_{x \in \mathcal{B}(G,F)} V_\pi^{G_{x,0+}}$, where $V_\pi^{G_{x,0+}} := \{v \in V_\pi \,|\, \pi(k)(v) = v \;\forall\, k \in G_{x,0+}\}$. This latter condition generalizes to non-irreducible representations and to representations over more general coefficient rings.

Definition 2.3 (Depth-zero representation with more general coefficients). *Let R be a $\mathbb{Z}[1/p]$-algebra. A smooth R-representation $\pi : G(F) \to \mathrm{Aut}_R(V_\pi)$ for an R-module V_π is said to have depth zero if $V_\pi = \sum_{x \in \mathcal{B}(G,F)} V_\pi^{G_{x,0+}}$.*

3 Complex representations of p-adic groups.

We keep the previous set-up, i.e., F is a non-archimedean local field, and G is a connected reductive group over F that splits over a tame extension. In this section we study the category of smooth complex representations of $G(F)$ and when we write "representation" in this section without specifying the coefficients, we mean a "smooth complex representation".

The building blocks of all smooth complex representations are the cuspidal representations, which are often also called *supercuspidal* representations in the setting of complex representations. These are the representations that are not subrepresentations of any parabolic induction from a proper parabolic subgroup of $G(F)$.[2] For a brief introduction to parabolic induction see [Fin25, §2.2] and for more details [Fin23, §2].

If p does not divide the order of the absolute Weyl group of G, then we have an explicit construction of all supercuspidal representations, see [Fin25, §2.6] for a brief overview and [Fin, §3 and §4] for a few more details.

Our goal is now to understand the structure of the whole category of all smooth representations of $G(F)$.

3.1 Bernstein decomposition.

Let $\pi : G(F) \to \mathrm{Aut}_{\mathbb{C}}(V)$ be an irreducible (smooth complex) representation of $G(F)$. Then one can show that there exists a parabolic subgroup $P \subseteq G$ with Levi subgroup M and a supercuspidal irreducible representation σ of $M(F)$ such that π is contained in the parabolic induction $\mathrm{Ind}_{P(F)}^{G(F)} \sigma$. If $P' \subseteq G$ is another parabolic subgroup with Levi subgroup M' and σ' a supercuspidal representation of $M'(F)$ such that $\pi \subseteq \mathrm{Ind}_{P'(F)}^{G'(F)} \sigma'$, then it turns out that there exists $g \in G(F)$ such that $M' = gMg^{-1}$ and $\sigma' \simeq {}^g\sigma$, where ${}^g\sigma$ is the representation of $M'(F)$ that satisfies ${}^g\sigma(m') = \sigma(g^{-1}m'g)$ for $m' \in M'(F)$. We call the $G(F)$-conjugacy class of the pair (M, σ) the *supercuspidal support* of π.

In order to decompose the category of all smooth representations of $G(F)$ we need to define a weaker equivalence class on the pairs consisting of Levi subgroups and supercuspidal representations.

Definition 3.1. *A character (i.e., a one dimensional representation) $\chi : G(F) \to \mathbb{C}^\times$ is called an* unramified character *if the restriction of χ to any compact subgroup of $G(F)$ is trivial.*

Definition 3.2. *Let M and M' be Levi subgroups of (parabolic subgroups of) G and let σ and σ' be supercuspidal representations of $M(F)$ and $M'(F)$, respectively. We say that (M, σ) and (M', σ') are* inertially equivalent *if and only if there exist $g \in G(F)$ and an unramified character χ of $M'(F)$ such that $M' = gMg^{-1}$ and $\sigma' \simeq {}^g\sigma \otimes \chi$.*

We denote the inertial equivalence by $\sim$, write $[M, \sigma]_G$ for the inertial equivalence class of the pair (M, σ), and denote by $\mathfrak{I}(G)$ the set of inertial equivalence classes, i.e., $\mathfrak{I}(G) = \{[M, \sigma]_G\}$ where M runs over the Levi subgroups of G and σ is a supercuspidal representation of $M(F)$. We might simply write $[M, \sigma]$ instead of $[M, \sigma]_G$ if the group G is clear from the context.

Let $[M, \sigma] \in \mathfrak{I}(G)$. Then we denote by $\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}$ the full subcategory of the category of smooth complex representations $\mathrm{Rep}_{\mathbb{C}}(G)$ of $G(F)$ whose objects are the following: A representation π of $G(F)$ is contained in $\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}$ if and only if for every irreducible subquotient π' of π, there exists a parabolic subgroup $P' \subseteq G$ with Levi subgroup M' and a supercuspidal representation σ' of $M'(F)$ with $(M', \sigma') \in [M, \sigma]$ such that $\pi' \hookrightarrow \mathrm{Ind}_{P'(F)}^{G'(F)} \sigma'$.

[2]In the case where $\mathcal{C}$ is an algebraically closed field of characteristic ℓ, cuspidal representations are those representations that are not subrepresentations of a proper parabolic induction and supercuspidal representations are those representations that are not subquotients of a proper parabolic induction. If $\mathcal{C} = \mathbb{C}$, then these two notions are equivalent and the term "supercuspidal" is more commonly used nowadays.

Theorem 3.3 (Bernstein [Ber84]). *We have an equivalence of categories*

$$\mathrm{Rep}_{\mathbb{C}}(G) \simeq \prod_{[M,\sigma]\in\mathfrak{I}(G)} \mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]},$$

and each full subcategory $\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}$ *is indecomposable.*

The above equivalence of categories is called the *Bernstein decomposition* and the full subcategory $\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}$ is called a *Bernstein block*.

3.2 Types. The structure of the Bernstein blocks can be analyzed via type theory that was introduced by Bushnell and Kutzko [BK98].

Definition 3.4. *Let* $[M,\sigma] \in \mathfrak{I}(G)$. *A pair* (K,ρ) *consisting of a compact, open subgroup* K *of* $G(F)$ *and an irreducible smooth complex representation* ρ *of* K *is an* $[M,\sigma]$-type *if the following property holds: For every irreducible (smooth complex) representation* π *of* $G(F)$ *the following are equivalent:*

1. π *is an object in* $\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}$,
2. ρ *is a subrepresentation of the restriction* $\pi|_K$ *of* π *to* K *(i.e.,* $\mathrm{Hom}_K(\rho,\pi) \neq \{0\}$*).*

We provide two very different examples of types.

If G is a simply-connected semisimple group, $M = G$ and $\sigma = \text{c-ind}_K^{G(F)}\rho$ is a supercuspidal representation that is induced from a compact, open subgroup K, then the pair (K,ρ) is a type for $\mathrm{Rep}_{\mathbb{C}}(G)^{[G,\sigma]}$, and the objects in $\mathrm{Rep}_{\mathbb{C}}(G)^{[G,\sigma]}$ are arbitrary (also infinite) direct sums of σ.

If G is a split reductive group with split maximal torus T and $\mathrm{Iw} := G_{x,0} \subset G(F)$ for x contained in a maximal facet of the apartment of T, then $(\mathrm{Iw}, \mathrm{triv})$ is a type for $\mathrm{Rep}_{\mathbb{C}}(G)^{[T,\mathrm{triv}]}$, where triv denotes the one-dimensional trivial representation. The group Iw is called *Iwahori subgroup*, and if $G(F) = \mathrm{SL}_2(\mathbb{Q}_p)$, for example, then (up to conjugation) $\mathrm{Iw} = \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}_{\det=1}$ for $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{Q}_p^\times \right\}$. The Bernstein block $\mathrm{Rep}_{\mathbb{C}}(G)^{[T,\mathrm{triv}]}$ is called the *principal block*. The principal block contains the trivial representation of $G(F)$.

3.3 Hecke algebras. A reason for the importance of types is that they lead to explicit algebras, called Hecke algebras, such that the Bernstein blocks are equivalent to the categories of modules over these algebras, see Theorem 3.7. To introduce Hecke algebras, let K be a compact, open subgroup of $G(F)$, and let $\rho : G(F) \to \mathrm{Aut}_{\mathbb{C}}(V_\rho)$ be an irreducible representation of K.

Definition 3.5. *The* Hecke algebra $\mathcal{H}(G,K,\rho)$ *is the* $\mathbb{C}$*-vector space of functions* $f : G(F) \to \mathrm{End}_{\mathbb{C}}(V_\rho)$ *satisfying*

1. $f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2)$ *for all* $k_1, k_2 \in K, g \in G(F)$*, and*
2. *the support of* f *is compact*

together with the multiplication given by the convolution *defined by*

$$(f_1 * f_2)(g) = \sum_{x \in G(F)/K} f_1(x) f_2(x^{-1}g)$$

for all $f_1, f_2 \in \mathcal{H}(G,K,\rho)$ *and* $g \in G(F)$.

Here $\mathrm{End}_{\mathbb{C}}(V_\rho)$ denotes the $\mathbb{C}$-linear endomorphisms of the $\mathbb{C}$-vector space V_ρ, i.e., the endomorphisms are not required to preserve the action of K. Note that

$$\sum_{x \in G(F)/K} f_1(x) f_2(x^{-1}g) = \int_{G(F)} f_1(x) f_2(x^{-1}g) dx$$

if we choose the measure dx to be the Haar measure that satisfies $\int_K 1 dx = 1$.

FACT 3.6. *We have an isomorphism of* $\mathbb{C}$*-algebras* $\mathcal{H}(G,K,\rho) \simeq \mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)} V_\rho\right)$ *where the product structure on the latter is given by composition.*

THEOREM 3.7 (Bushnell–Kutzko [BK98]). *If* (K,ρ) *is an* $[M,\sigma]$*-type, then the Bernstein block* $\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}$ *is equivalent to the category of right unital* $\mathcal{H}(G,K,\rho)$*-modules, i.e.,*

$$\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]} \simeq \textit{Mod-}\mathcal{H}(G,K,\rho).$$

The equivalence in the above theorem is given by sending $\pi \in \mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]}$ to the nontrivial vector space $\mathrm{Hom}_K(V_\sigma, V_\pi)$. The action of $\mathcal{H}(G,K,\rho) \simeq \mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)} V_\rho\right)$ on this space is given by using Frobenius reciprocity to identify

$$\mathrm{Hom}_K(V_\rho, V_\pi) \simeq \mathrm{Hom}_{G(F)}\left(\text{c-ind}_K^{G(F)} V_\rho, V_\pi\right),$$

and $\mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)} V_\rho\right)$ acts on the right hand side via precomposition.

Of course this result is only of use if

- we know that types exist for the Bernstein blocks that we want to study, and
- we understand the structure of the resulting Hecke algebras.

The first concern will be answered in Sections 3.5 and 3.6, and the second question will be discussed in Sections 3.4 and 3.7 below.

3.4 First examples of Hecke algebras. We provide three examples of Hecke algebras attached to types. We will later see that in general the structure of Hecke algebras of types is a combination of generalizations of the examples below, see Theorem 3.15.

3.4.1 A trivial Hecke algebra. We start with the examples of types provided in Section 3.2. Let G be a simply-connected semisimple group and let (K,ρ) be such that $\text{c-ind}_K^{G(F)} V_\rho$ is irreducible, supercuspidal. Then using Fact 3.6 and Schur's lemma, we have $\mathcal{H}(G,K,\rho) \simeq \mathrm{End}_{G(F)}(\text{c-ind}_K^{G(F)} V_\rho) \simeq \mathbb{C}$. We see that indeed the category of $\mathbb{C}$-modules, i.e., complex vector spaces, is equivalent to the Bernstein block $\mathrm{Rep}_{\mathbb{C}}(G)^{[G,\sigma]}$ described in Section 3.2.

3.4.2 The Iwahori–Hecke algebra of SL_2**.** To provide a first non-trivial example of a Hecke algebra $\mathcal{H}(G,K,\rho)$ we consider the case $G = \mathrm{SL}_2$ and $(K,\rho) = (\mathrm{Iw}, \mathrm{triv})$, which is a type for the Bernstein block $\mathrm{Rep}_{\mathbb{C}}(\mathrm{SL}_2)^{[T,\mathrm{triv}]}$ as mentioned in Section 3.2. By definition, as a vector space, we have

$$\mathcal{H}(\mathrm{SL}_2, \mathrm{Iw}, \mathrm{triv}) = \{\mathrm{Iw} \setminus \mathrm{SL}_2(F)/\mathrm{Iw} \to \mathbb{C}\}.$$

Let $N(T)$ denote the normalizer of T in G and $\mathcal{O}$ the ring of integers in F, then we have an isomorphism of sets $\mathrm{Iw} \setminus G(F)/\mathrm{Iw} \simeq N(T)(F)/T(\mathcal{O})$ arising from the embedding $N(T)(F) \hookrightarrow G(F)$. Note that $N(T)(F)/T(\mathcal{O})$ is actually a group and is isomorphic to the affine Weyl group $W_{\mathrm{aff}}(\widetilde{A}_1) = \left\langle s_0, s_1 \mid s_0^2 = s_1^2 = 1\right\rangle$ of type $\widetilde{A}_1$. Thus, as a vector space, $\mathcal{H}(\mathrm{SL}_2, \mathrm{Iw}, \mathrm{triv}) = \oplus_{w \in W_{\mathrm{aff}}(\widetilde{A}_1)} \mathbb{C} \cdot \mathbb{T}_w$, where $\mathbb{T}_w$ denotes a basis element of the vector space. It remains to understand the multiplication arising from convolution that turns the vector space into an algebra. This structure is generated by asking $\mathbb{T}_1$ to be a multiplicative unit and by the following two relations, where q denotes the cardinality of the residue field of F:

$$\begin{aligned} \mathbb{T}_{s_i} \cdot \mathbb{T}_{s_i} &= (q-1)\mathbb{T}_{s_i} + q\mathbb{T}_1 && \text{for } i \in \{0,1\}, \\ \mathbb{T}_w &= \mathbb{T}_{s_{i_1}} \cdot \mathbb{T}_{s_{i_2}} \cdot \ldots \cdot \mathbb{T}_{s_{i_n}} && \text{for } w = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_n} \text{ a reduced expression with } i_j \in \{0,1\}. \end{aligned}$$

Note that the last relation in particular means that $\mathbb{T}_w = \mathbb{T}_{s_{i_1}} \cdot \mathbb{T}_{s_{i_2}} \cdot \ldots \cdot \mathbb{T}_{s_{i_n}}$ does not depend on the choice of reduced expression $s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_n}$ for w.

3.4.3 The Hecke algebra of GL_1**.** Let $G = \mathrm{GL}_1$, $K = \mathrm{GL}_1(\mathcal{O}) = \mathcal{O}^\times$ for $\mathcal{O}$ the ring of integers in F, $\rho = \mathrm{triv}$. Then, as a vector space, $\mathcal{H}(\mathrm{GL}_1, \mathcal{O}^\times, \mathrm{triv}) = \{F^\times/\mathcal{O}^\times \to \mathbb{C}\}$, and one can check that the algebra structure is the one of a group algebra of $\mathbb{Z}$, i.e., that $\mathcal{H}(\mathrm{GL}_1, \mathcal{O}^\times, \mathrm{triv}) \simeq \mathbb{C}[\mathbb{Z}]$ as $\mathbb{C}$-algebras. Some authors might also write $\mathbb{C}[t, t^{-1}]$ instead of $\mathbb{C}[\mathbb{Z}]$.

3.5 Depth-zero types. In order to use the theory of types to study Bernstein blocks, we need to know that types exist, and ideally we would like to have an explicit construction for them. We start with the special case of Bernstein blocks that consist of depth-zero representations, generalizing the construction of depth-zero supercuspidal representations.

Let $x \in \mathcal{B}(G, F)$. Then Moy and Prasad ([MP96, §6.3]) construct a Levi subgroup $M \subseteq G$ with the properties that $x \in \mathcal{B}(M, F) \subseteq \mathcal{B}(G, F)$, that x is contained in a facet of minimal dimension of $\mathcal{B}(M, F)$, and that the inclusion $M_{x,0} \hookrightarrow G_{x,0}$ induces an isomorphism $M_{x,0}/M_{x,0+} \xrightarrow{\simeq} G_{x,0}/G_{x,0+}$. Hence we also have

$$(G_{x,0} \cdot M_x)/G_{x,0+} \simeq M_x/M_{x,0+},$$

where M_x denotes the stabilizer of x in $M(F)$, which is a compact, open subgroup of $M(F)$.

Proposition 3.8 (Moy–Prasad [MP96], see also Kim–Yu [KY17, 7.1]). *Let ρ^0 be an irreducible representation of $G_{x,0}M_x$ that is trivial on $G_{x,0+}$ and such that $\rho^0|_{G_{x,0}}$ is a cuspidal representation of the finite group of Lie type $G_{x,0}/G_{x,0+} \simeq M_{x,0}/M_{x,0+}$. Then the pair $(G_{x,0}M_x, \rho^0)$ is a type for a Bernstein block that consists of depth-zero representations.*

Moreover, every Bernstein block consisting of depth-zero representations admits a type of this form.

We may refer to $(G_{x,0}M_x, \rho^0)$ as a *depth-zero type* and to the corresponding block as *depth-zero Bernstein block*. Note that if a Bernstein block contains a depth-zero representation, then all its representations are of depth zero.

3.6 Types constructed by Kim and Yu – with a twist. Based on the construction of depth-zero types, we will now indicate a construction of types for Bernstein blocks consisting of representations of positive depth, based on the work of Kim and Yu [KY17] that is based on Yu's construction of supercuspidal representations [Yu01], but twisted by a quadratic character arising from the work of Fintzen, Kaletha and Spice [FKS23] as in Adler–Fintzen–Mishra–Ohara [AFMOb].

To simplify the input for the construction of types for this paper, we assume that p does not divide the order of the absolute Weyl group of G. Then we say that a *shortened G-datum* is a triple $(G^0 \subseteq G, (K^0, \rho^0), \phi)$, where

- $G^0 \subseteq G$ is a tame twisted Levi subgroup, i.e., a subgroup of G such that there exists a tamely ramified field extension E of F so that the base change G^0_E is a Levi subgroup of a parabolic subgroup of G_E,
- (K^0, ρ^0) is a depth-zero type for G^0 as in Proposition 3.8 with $K^0 = G^0_{x,0}M^0_x$ for an appropriate point $x \in \mathcal{B}(G^0, F)$ and attached Levi subgroup $M^0 \subseteq G^0$,
- ϕ is a *sufficiently generic* character of $G^0(F)$, i.e., ϕ is a character of $G^0(F)$ such that if $T^0 \subset G^0$ is a maximal torus of G^0 that splits over a tame extension E (by our assumption all maximal tori split over a tame extension), then for every $\alpha \in \Phi(G, T^0) \smallsetminus \Phi(G^0, T^0)$ we have

  $$\phi(\mathrm{Nm}_{E/F}(\check{\alpha}(\varpi_E \mathcal{O}_E))) \neq 1.$$

  Here $\Phi(X, T^0)$ denotes the roots of X with respect to T^0, $\mathrm{Nm}_{E/F}$ denotes the norm for the extension E/F, the ring $\mathcal{O}_E$ is the ring of integers of E and ϖ_E is a uniformizer of $\mathcal{O}_E$.

Moreover, the point x needs to satisfy a genericity condition, which is satisfied away from a discrete set of hyperplanes. The interested reader can find the details in [KY17, 3.2, 3.5. and 7.2(D2)].

Remark 3.9. We caution the reader that the notions of "shortened G-datum" and "sufficiently generic character" have been introduced for this survey article to present the construction of types in a simpler way and more aligned with Section 4. Traditionally, the input for the construction of Kim and Yu is a *G-datum* that consists of a sequence of twisted Levi subgroups, $G^0 \subsetneq G^1 \subsetneq G^2 \subsetneq \ldots \subsetneq G^d = G$, a depth-zero type as above, a point in the Bruhat–Tits building of G^0 as above, a sequence of positive real numbers, $0 < r_0 < r_1 < \ldots r_{d-1} \leq r_d$, and a sequence of characters, one on each group of the twisted Levi sequence, each of which satisfies a genericity condition, see [KY17, 7.2] and [AFMOb, Definition 4.1.1] for details. The product of the restriction of these characters to $G^0(F)$ is the character ϕ in our shortened input. We have made the assumption that p does not divide the order of the absolute Weyl group of G, so that the above property of "sufficiently generic" allows one to obtain from ϕ a sequence of twisted Levi subgroups and generic characters thereof whose product is ϕ, see [Kal19, 3.6], where this process is called "Howe factorization". Note also that in this article we assumed for simplicity that our depth-zero input satisfies $K^0 = G^0_{x,0} \cdot M^0_x$ while [KY17] and [AFMOb] allow for a more general choice for K^0, which leads to types that describe a finite union of Bernstein blocks rather than a single Bernstein block.

Let $(G^0 \subseteq G, (K^0, \rho^0), \phi)$ be a shortened G-datum. From this shortened G-datum one can construct (following Kim and Yu [KY17], but including a twist by a quadratic character arising from Fintzen–Kaletha–Spice [FKS23] as carried out by Adler–Fintzen–Mishra–Ohara [AFMOb]) a pair (K, ρ) consisting of a compact open subgroup $K \subseteq G(F)$ that satisfies $K = K^0 \cdot K_{0+}$ for a pro-p group $K_{0+} \subset G(F)$ that is normalized by K^0, and a representation ρ of K of the following shape:

$$\rho = \rho^0 \otimes \kappa^{\mathrm{nt}} \otimes \epsilon = \rho^0 \otimes \kappa,$$

where ρ^0 also denotes the extension of ρ^0 to K that is trivial on K_{0+}, the representation κ^{nt} is constructed from the character ϕ via the theory of Heisenberg–Weil representations, the representation ϵ is a quadratic character arising from [FKS23] that is trivial on K_{0+}, and $\kappa := \epsilon\kappa^{\mathrm{nt}}$. See [AFMOb, §4.1] for more details on the construction of κ. Denote by N the kernel of κ restricted to K_{0+} and let K_+ be the preimage of the center of K_{0+}/N. Then κ restricted to K_+ acts via a character $\hat{\phi}$ times the identity, and we have $\hat{\phi}|_{G^0(F)\cap K_+} = \phi|_{G^0(F)\cap K_+}$. Moreover, $G^0(F) \cap K_+ = G^0(F)_{x,0+}$. (For the reader who prefers to work with the standard input for Kim and Yu's construction of types, the group K_{0+} is given by $G^0_{x,0+}G^1_{x,r_0/2}G^2_{x,r_1/2}\dots G^d_{x,r_{d-1}/2}$, and the group K_+ is the normal subgroup $K_+ = G^0_{x,0+}G^1_{x,(r_0/2)+}G^2_{x,(r_1/2)+}\dots G^d_{x,(r_{d-1}/2)+}$.)

We write M for the centralizer $\mathrm{Cent}_G(Z_{\mathrm{split}}(M^0))$ in G of the maximal split torus $Z_{\mathrm{split}}(M^0)$ in the center of M^0. Then M is a Levi subgroup of a parabolic subgroup of G and we obtain the following result.

THEOREM 3.10 (Kim–Yu [KY17] (and Fintzen [Fin21a] or Fintzen–Kaletha–Spice [FKS23])). *The pair (K, ρ) is an $[M, \sigma]$-type (for some supercuspidal representation σ of $M(F)$).*

Under our above tameness assumption this construction provides us with types for every Bernstein block.

THEOREM 3.11 (Kim–Yu [KY17] based on Kim [Kim07] (for F of characteristic zero and p very large) and Fintzen [Fin21b] (the general case)). *Recall that we assume that G splits over a tamely ramified field extension of F and that p does not divide the order of the absolute Weyl group of G. Then for every $[M, \sigma] \in \mathfrak{I}(G)$, there exists a shortened G-datum whose associated pair (K, ρ) by the construction above is an $[M, \sigma]$-type.*

3.7 Structure of Hecke algebras and reduction to depth zero. From now on let $(G^0 \subseteq G, (K^0, \rho^0), \phi)$ be a shortened G-datum, and let (K, ρ) be the corresponding $[M, \sigma]$-type from Section 3.6. Recall that (K^0, ρ^0) is an $[M^0, \sigma^0]_{G^0}$-type for some depth-zero supercuspidal representation σ^0 of $M^0(F)$. In this section we will discuss an isomorphism between $\mathcal{H}(G, K, \rho)$ and $\mathcal{H}(G, K^0, \rho^0)$ and use it to describe the structure of $\mathcal{H}(G, K, \rho)$, following Adler–Fintzen–Mishra–Ohara [AFMOa, AFMOb].

In order to describe the structure of the Hecke algebras, we first introduce some more notation in line with [AFMOb]. We set

$$\begin{aligned} K_{M^0} &:= K \cap M^0(F) = K^0 \cap M^0(F) = M^0_x, \\ \rho_{M^0} &:= \rho^0|_{K_{M^0}}, \\ N(\rho_{M^0})_{[x]_{M^0}} &:= \{n \in G^0(F) \mid nM^0n^{-1} = M^0,\, nK^0_M n^{-1} = K^0_M,\, {}^n\rho_{M^0} \simeq \rho_{M^0}\}. \end{aligned}$$

Our isomorphism between the two Hecke algebras $\mathcal{H}(G, K, \rho)$ and $\mathcal{H}(G, K^0, \rho^0)$ will be constructed in a support-preserving way. To make sense of such a statement, we first observe the following structure of their supports.

FACT 3.12. *We have* $\mathrm{Supp}\big(\mathcal{H}(G, K, \rho)\big) = K \cdot \mathrm{Supp}\big(\mathcal{H}(G^0, K^0, \rho^0)\big) \cdot K = K \cdot N(\rho_{M^0})_{[x]_{M^0}} \cdot K.$

Moreover, if we restrict the support to a single double coset, then we have the following result.

FACT 3.13. *Let $g \in \mathrm{Supp}(\mathcal{H}(G, K, \rho))$. The $\mathbb{C}$-subspace of functions in $\mathcal{H}(G, K, \rho)$ that are supported on KgK has dimension one.*

While the support is a priori only a set of double cosets, the next result allows us to endow it with a group structure.

PROPOSITION 3.14 (Adler–Fintzen–Mishra–Ohara [AFMOa, AFMOb]). *The inclusion map induces a bijection*

$$N(\rho_{M^0})_{[x]_{M^0}}/(N(\rho_{M^0})_{[x]_{M^0}} \cap K_{M^0}) \xrightarrow{\simeq} K\backslash \mathrm{Supp}\big(\mathcal{H}(G, K, \rho)\big)/K.$$

For example, if $G = G^0 = \mathrm{SL}_2$, $K^0 = K = \mathrm{Iw}$ and $\rho^0 = \rho = \mathrm{triv}$ as in Section 3.4.2, then $M^0 = T$, $K_{M^0} = T(\mathcal{O})$, $N(\rho_{M^0})_{[x]_{M^0}} = N(T)(F)$ and $N(\rho_{M^0})_{[x]_{M^0}} \cap K_{M^0} = T(\mathcal{O})$. Since the intersection $N(\rho_{M^0})_{[x]_{M^0}} \cap K_{M^0}$ is a normal subgroup of $N(\rho_{M^0})_{[x]_{M^0}}$, the quotient $N(\rho_{M^0})_{[x]_{M^0}}/(N(\rho_{M^0})_{[x]_{M^0}} \cap K_{M^0})$ is a group, which we denote by $W^\heartsuit$. This equips the support of our Hecke algebras with a group structure. In the previous example, $W^\heartsuit = N(T)(F)/T(\mathcal{O}) \simeq W_{\mathrm{aff}}(\widetilde{A}_1)$.

In order to describe the Hecke algebra structure using $W^\heartsuit$ we recall a few common algebra structures. We start with the notion of a general affine Hecke algebra that generalizes the affine Hecke algebra that we have encountered in Section 3.4.2. Let W_{aff} be an affine Weyl group with set of simple reflections S_{aff}. For each $s \in S_{\mathrm{aff}}$, we let q_s be a complex number such that $q_s = q_{s'}$ if s and s' are conjugate. Then the affine Hecke algebra $\mathcal{H}_{\mathrm{aff}}(W_{\mathrm{aff}}, \{q_s\})$ is as a complex vector space $\oplus_{w \in W_{\mathrm{aff}}} \mathbb{C} \cdot \mathbb{T}_w$ with the relations for the multiplication structure generated by asking $\mathbb{T}_1$ to be the multiplicative identity and:

$$\mathbb{T}_s \cdot \mathbb{T}_s = (q_s - 1)\mathbb{T}_s + q_s \mathbb{T}_1 \text{ for } s \in S_{\mathrm{aff}}$$
$$\mathbb{T}_w = \mathbb{T}_{s_1} \cdot \mathbb{T}_{s_2} \cdot \ldots \cdot \mathbb{T}_{s_n} \text{ for } w = s_1 \cdot s_2 \cdot \ldots \cdot s_n \text{ a reduced expression with } s_i \in S_{\mathrm{aff}}.$$

For a group Ω and a 2-cocycle $\mu : \Omega \times \Omega \to \mathbb{C}^\times$, we denote by $\mathbb{C}[\Omega, \mu]$ the corresponding twisted group algebra, i.e., $\mathbb{C}[\Omega, \mu] \simeq \bigoplus_{t \in \Omega} \mathbb{C} \cdot b_t$ as a vector space with multiplication given by $b_{t_1} b_{t_2} = \mu(t_1, t_2) b_{t_1 t_2}$. We assume that $\mu(1, t) = \mu(t, 1) = 1$ so that b_1 is a multiplicative identity.

If Ω acts on W_{aff} such that $\omega(s_i) = s_j$ with $q_{s_j} = q_{s_i}$ for all $\omega \in \Omega, s_i \in S_{\mathrm{aff}}$, then we can form the semi-direct product algebra $\mathbb{C}[\Omega, \mu] \ltimes \mathcal{H}_{\mathrm{aff}}(W_{\mathrm{aff}}, \{q_s\})$ as follows. As a vector space $\mathbb{C}[\Omega, \mu] \ltimes \mathcal{H}_{\mathrm{aff}}(W_{\mathrm{aff}}, \{q_s\})$ is given by $\mathbb{C}[\Omega, \mu] \otimes_{\mathbb{C}} \mathcal{H}_{\mathrm{aff}}(W_{\mathrm{aff}}, \{q_s\})$ and the multiplication is given by asking $\mathcal{H}_{\mathrm{aff}}(W_{\mathrm{aff}}, \{q_s\}) \simeq b_1 \otimes \mathcal{H}_{\mathrm{aff}}(W_{\mathrm{aff}}, \{q_s\})$ and $\mathbb{C}[\Omega, \mu] \simeq \mathbb{C}[\Omega, \mu] \otimes \mathbb{T}_1$ to be subalgebras and for $t \in \Omega$ and $w \in W_{\mathrm{aff}}$ we have $b_t \cdot \mathbb{T}_w = \mathbb{T}_{twt^{-1}} \cdot b_t$. Then it turns out that all the Hecke algebras arising from the types constructed above are of this form.

THEOREM 3.15 (Adler–Fintzen–Mishra–Ohara [AFMOb, Theorem 4.4.1]). *The group $W^\heartsuit$ admits the structure of a semi-direct product $W^\heartsuit \simeq \Omega(\rho_{M^0}) \ltimes W(\rho_{M^0})_{\mathrm{aff}}$ where $W(\rho_{M^0})_{\mathrm{aff}}$ is an affine Weyl group with a set of simple reflections $S(\rho_{M^0})_{\mathrm{aff}}$ that is preserved under the action of $\Omega(\rho_{M^0})$ and such that we have an isomorphism of algebras*

$$\mathcal{H}(G, K, \rho) \simeq \mathbb{C}[\Omega(\rho_{M^0}), \mu] \ltimes \mathcal{H}_{\mathrm{aff}}(W(\rho_{M^0})_{\mathrm{aff}}, \{q_s\}) \tag{3.1}$$

for some 2-cocycle $\mu : \Omega(\rho_{M^0}) \times \Omega(\rho_{M^0}) \to \mathbb{C}^\times$ and some explicitly described $q_s \in \mathbb{Q}_{>1}$ for $s \in S(\rho_{M^0})_{\mathrm{aff}}$ with $q_{\omega s \omega^{-1}} = q_s$ for all $\omega \in \Omega(\rho_{M^0}), s \in S(\rho_{M^0})_{\mathrm{aff}}$.

For a more detailed description of μ and q_s see [AFMOb, Theorem 4.4.1]. In order to show that $\mathcal{H}(G, K, \rho)$ and $\mathcal{H}(G^0, K^0, \rho^0)$ are isomorphic it suffices to show that the right hand sides of (3.1) agree, whose proof in [AFMOa, AFMOb] is intertwined with proving the structural result, Theorem 3.15, for $\mathcal{H}(G, K, \rho)$ itself.

This isomorphism can be made explicit as follows.

THEOREM 3.16 (Adler–Fintzen–Mishra–Ohara [AFMOb, Theorem 4.3.11] and [AFMOa, Theorem 4.4.11]). *There exists a representation $\widetilde{\kappa} : N(\rho_{M^0})_{[x]_{M^0}} \cdot (K \cap M(F)) \to \mathrm{End}(V_\kappa)$ such that $\widetilde{\kappa}|_{K \cap M(F)} = \kappa|_{K \cap M(F)}$ and such that there exists an algebra-isomorphism*

$$\mathcal{I} : \mathcal{H}(G^0, K^0, \rho^0) \xrightarrow{\simeq} \mathcal{H}(G, K, \rho)$$

defined by the following:

If $\varphi \in \mathcal{H}(G^0, K^0, \rho^0)$ is supported on $K^0 n K^0$ with $n \in N(\rho_{M^0})_{[x]_{M^0}}$, then $\mathcal{I}(\varphi)$ is supported on KnK and

$$\mathcal{I}(\varphi)(n) = d_n \cdot \varphi(n) \otimes \widetilde{\kappa}(n) \quad \text{ with } \quad d_n = \sqrt{\frac{|K^0/(nK^0 n^{-1} \cap K^0)|}{|K/(nKn^{-1} \cap K)|}}.$$

Remark 3.17. In order to obtain Theorem 3.16 it is crucial that one works with $\rho = \rho^0 \otimes \kappa$ and not with $\rho^0 \otimes \kappa^{\mathrm{nt}}$. Replacing $\kappa = \kappa^{\mathrm{nt}} \otimes \epsilon$ by κ^{nt}, the resulting Hecke algebras are not always isomorphic, see [AFMOb, A.2] for an example.

Combining Theorem 3.16 with Theorem 3.11 and Theorem 3.7 we obtain the following corollary.

Corollary 3.18 (Adler–Fintzen–Mishra–Ohara [AFMOb, Theorem 4.5.2]). *Recall that we assume that G splits over a tamely ramified extension and that p does not divide the order of the absolute Weyl group of G. Let $[M,\sigma] \in \mathfrak{I}(G)$. Then there exists a tamely ramified twisted Levi subgroup $G^0 \subseteq G$ and $[M^0,\sigma^0] \in \mathfrak{I}(G^0)$ with σ^0 of depth zero such that*

$$\mathrm{Rep}_{\mathbb{C}}(G)^{[M,\sigma]} \simeq \mathrm{Rep}_{\mathbb{C}}(G^0)^{[M^0,\sigma^0]}.$$

3.8 Some prior work. Already in the 1960s Iwahori and Matsumoto [IM65] described the Hecke algebras for the special case that G is an adjoint, split semisimple group and $(K,\rho) = (\mathrm{Iw}, \mathrm{triv})$. This was later generalized by Goldstein to the case that G is split semisimple and $(K,\rho) = (\mathrm{Iw}, \chi)$ for a depth-zero character χ, and in 1993, Morris [Mor93] provided a description of the Hecke algebras attached to pairs $(G_{x,0}, \rho^0)$ for ρ^0 an appropriate depth-zero representation as in Theorem 3.15. This means, if G is simply connected semisimple, Morris provided a description of the Hecke algebras that describe a depth-zero Bernstein block. In general, the Hecke algebras that Morris described correspond to a finite union of depth-zero Bernstein blocks and his result was generalized in [AFMOa, §5] to describe all the Hecke algebras of all depth-zero Bernstein blocks.

Beyond the depth-zero setting, Hecke algebras of types have been described for various special classes of groups or special classes of types. To mention a few general results, in 1993 Bushnell and Kutzko [BK93] provided a description of the Hecke algebra for all Bernstein blocks for the group GL_n that also shows the equivalence between arbitrary blocks and depth-zero blocks. Based on their results, Goldberg and Roche [GR02, GR05] treated the case of SL_n, and Sécherre and Stevens [SS08] the case of inner forms of GL_n. For general split reductive groups, Roche [Roc98] described in the 1990s the Hecke algebras attached to principal series Bernstein blocks assuming p is not too small and F is of characteristic zero, i.e., the case where M is a split maximal torus of G. Roche also obtained an equivalence between principal series Bernstein blocks of arbitrary depth and blocks of depth-zero. More recently, Ohara [Oha24] obtained a reduction-to-depth-zero result in the other extreme case, the case of $M = G$, i.e., for supercuspidal Bernstein blocks.

3.9 A more general setting. While we have been working with complex coefficients and under a tameness assumption throughout Section 3, the isomorphisms of the Hecke algebras, Theorems 3.15 and 3.16, are obtained in Adler–Fintzen–Mishra–Ohara [AFMOa, AFMOb] in a much more general setting, including when considering a representation ρ with $\mathcal{C}$-coefficients for an algebraically closed field $\mathcal{C}$ of characteristic different from p. This allows one to draw conclusion about irreducible $\mathcal{C}$-representations of p-adic groups, but it does not provide an equivalence of categories as in Corollary 3.18 in general because the Bernstein decomposition and the related type theory do not work in general. Therefore we have to use very different techniques to understand the whole category of smooth $\mathcal{C}$-representations, which is the subject of Section 4.

4 Representations of p-adic groups with more general coefficients. We keep the notation from above, i.e., F is a non-archimedean local field, G a connected reductive group over F that splits over a tamely ramified extension of F. We also assume throughout this section that p does not divide the order of the absolute Weyl group of G.

In this section, we study smooth R-representations of $G(F)$, where R denotes any $\mathbb{Z}[\mu_{4p^\infty}, 1/\sqrt{p}]$-algebra. To give some examples for possible coefficient rings, R could be the complex numbers $\mathbb{C}$ as in the previous section, but R could also be $\bar{\mathbb{F}}_\ell$ for $\ell \neq p$, or R could be the ring $\bar{\mathbb{Z}}[1/p]$ or the ring $\mathbb{Z}[\mu_{4p^\infty}, 1/\sqrt{p}]$ itself.

Working with such a much more general coefficient ring R leads to several complications. Already when working with $\bar{\mathbb{F}}_\ell$-coefficients, the Bernstein decomposition that we saw in Theorem 3.3 no longer holds in general. On the one hand, the notion of supercuspidal support is no longer well defined in general, see Dat [Dat18b] and Dudas [Dud18] for an example. On the other hand, while there is still a notion of cuspidal support, this cuspidal support does not decompose the category into a product of subcategories, see Vigneras [Vig96, II.2.5] for an example. Relatedly, studying blocks via type theory and Hecke algebras as in Section 3 is no longer a tool that we have available. This means we need to use different techniques that we will outline in this section, and which allow us to obtain the following result (where we explain the notation below).

Theorem 4.1 (Dat–Fintzen [DF]). *We recall that we assume that p does not divide the order of the absolute Weyl group of G, and R is a $\mathbb{Z}[\mu_{4p^\infty}, 1/\sqrt{p}]$-algebra. Then we have equivalences of categories*

$$\mathrm{Rep}_R(G) \simeq \prod_{\{(\phi,\iota)\}/\sim} \mathrm{Rep}_R(G)^{[\phi,\iota]} \quad \text{and} \quad \mathrm{Rep}_R(G)^{[\phi,\iota]} \simeq \mathrm{Rep}_R(G_\iota)_{\text{depth-zero}},$$

where $\mathrm{Rep}_R(G_\iota)_{\text{depth-zero}}$ *denotes the full subcategory of the* $\mathrm{Rep}_R(G_\iota)$ *consisting of depth-zero representations.*

The definition of the pairs (ϕ, ι), the equivalence relation on them, the group G_ι, and the subcategory $\mathrm{Rep}_R(G)^{[\phi,\iota]}$ is the subject of the following subsections. For a more detailed theorem statement with more precise (slightly weaker) assumptions we refer the reader to the soon appearing work [DF].

Note that if $R = \bar{\mathbb{Z}}[1/p]$, then Dat and Lanard [DL25] have shown that all depth-zero representations together form a block, i.e., are not further decomposable as a product of smaller non-trivial subcategories. Hence, if $R = \bar{\mathbb{Z}}[1/p]$, then the above subcategories are all indecomposable, i.e., the above decomposition of $\mathrm{Rep}_{\bar{\mathbb{Z}}[1/p]}(G)$ is a block decomposition.

4.1 The indexing set – wild inertia parameter ϕ. Let W_F be the Weil group of F and P_F the wild inertia subgroup of W_F. We write $\Phi(G)$ for the set of relevant Langlands parameters for G, i.e., appropriate homomorphisms $W_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G} \rtimes W_F$ up to $\hat{G}$-conjugacy, where $\hat{G}$ denotes the complex dual group of G. We set

$$\Phi(P_F, G) := \{\varphi|_{P_F} \mid \varphi \in \Phi(G)\},$$

i.e., $\Phi(P_F, G)$ consists of morphisms $\phi : P_F \to \hat{G} \rtimes W_F$ up to $\hat{G}$-conjugation that can be extended to a relevant Langlands parameter φ. We call elements of $\Phi(P_F, G)$ *wild inertia parameters*. The element ϕ in Theorem 4.1 runs over (representatives for) elements of $\Phi(P_F, G)$.

4.2 The indexing set – Levi-center embedding ι. We fix $\phi \in [\phi] \in \Phi(P_F, G)$ and choose a representative $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G} \rtimes W_F$ for a relevant Langlands parameter whose restriction to P_F is ϕ. Then one can prove that our assumption on p implies that the centralizer $\hat{G}_\phi := \mathrm{Cent}_{\hat{G}}(\phi)$ of ϕ in the complex group $\hat{G}$ is a Levi subgroup of $\hat{G}$, in particular, it is connected. In this section, we attach to $\hat{G}_\phi \subseteq \hat{G}$ a twisted Levi subgroup $G_\iota \subseteq G$ (which will depend on a choice of ι that we are about to introduce). We write $(\hat{G}_\phi)_{\mathrm{ab}}$ for the cocenter of $\hat{G}_\phi$ and equip it with an action of W_F arising from the conjugation action of $\varphi(W_F)$ on $\hat{G}_\phi$. The resulting action of W_F on $(\hat{G}_\phi)_{\mathrm{ab}}$ does not depend on the choice φ, because if φ' is another representative for an element of $\Phi(G)$ whose restriction to P_F agrees with ϕ, then $((\varphi')^{-1} \circ \varphi)(g)$ commutes with $\phi(p)$ for all $g \in W_F$ and all $p \in P_F$, in particular, $(\varphi')^{-1} \circ \varphi$ factors through $\hat{G}_\phi$, and hence the action induced on $(\hat{G}_\phi)_{\mathrm{ab}}$ via conjugation is trivial. We write S_ϕ for the dual F-torus of $(\hat{G}_\phi)_{\mathrm{ab}}$.

Generalizing the duality of maximal tori embedding, we obtain from this set-up a canonical $G(\bar{F})$-conjugacy class of Levi-center embeddings $(S_\phi)_{\bar{F}} \hookrightarrow G_{\bar{F}}$, i.e., embeddings such that $\iota((S_\phi)_{\bar{F}})$ is the connected center of its centralizer, see [DF, 2.1] for details. We write $\mathcal{I}_\phi$ for the set of $G(F)$-conjugacy classes of embeddings that are contained in the above $G(\bar{F})$-conjugacy class of Levi-center embeddings and that are defined over F. The set $\mathcal{I}_\phi$ is the indexing set for (the equivalence classes of) ι in Theorem 4.1. For $\iota \in [\iota] \in \mathcal{I}_\phi$, we set $G_\iota := \mathrm{Cent}_G(\iota(S_\phi))$, which is a twisted Levi subgroup of G. The dual of G_ι turns out to be $\hat{G}_\phi$.

4.3 The Serre subcategories $\mathrm{Rep}_R(G)^{[\phi,\iota]}$. We fix $\phi \in [\phi] \in \Phi(P_F, G)$ and $\iota \in [\iota] \in \mathcal{I}_\phi$, where $[\phi]$ denotes the $\hat{G}$-conjugacy class of ϕ and $[\iota]$ denotes the $G(F)$-conjugacy class of ι. The construction of $\mathrm{Rep}_R(G)^{[\phi,\iota]}$ proceeds in four steps, which we sketch below. For the details see [DF].

Step 1. The first step consists of obtaining from ϕ a character of a compact, open subgroup of $G(F)$, i.e., to move from the Langlands parameter side to the representation theory of p-adic groups side. One can prove ([DF]) that one can choose an extension $\varphi : W_F \to \hat{G} \rtimes W_F$ of ϕ that factors through $Z(\hat{G}_\phi) \rtimes W_F$ for some (not necessarily standard) embedding of $Z(\hat{G}_\phi) \rtimes W_F$ into $\hat{G} \rtimes W_F$. Using the Langlands correspondence for characters by Borel [Bor79, 10.2] this leads to a character $\check{\varphi} : G_\iota(F) \to \mathbb{C}^\times$. We can choose φ such that $\check{\varphi}$ factors through μ_{p^∞} ([DF]). Hence, composing $\check{\varphi} : G_\iota(F) \to \mu_{p^\infty}$ with $\mu_{p^\infty} \to R^\times$ leads to an R-valued character of $G_\iota(F)$. It turns out that the character $\check{\varphi}$ of $G_\iota(F)$ is sufficiently generic, as defined in Section 3.6. Moreover, the restriction of $\check{\varphi}$ to $(G_\iota)_{x,0+} =: G_{\iota,x,0+}$ is independent of the choice of extension φ of ϕ ([DF]) for any $x \in \mathcal{B}(G_\iota, F)$.

Step 2. Let $x \in \mathcal{B}(G_\iota, F)$. Since $\check{\varphi}$ is a sufficiently generic character of the twisted Levi subgroup $G_\iota(F)$ as defined in Section 3.6, we obtain from the construction of types by Kim and Yu discussed in that section a compact, open subgroup $K_+ \supseteq G_{\iota,x,0+}$, which we call $K_{x,+}$ here[3] to record the point x, together with a character $\check{\phi}^+_{\iota,x}$ of $K_{x,+}$ that satisfies $\check{\phi}^+_{\iota,x}|_{G_{\iota,x,0+}} = \check{\varphi}|_{G_{\iota,x,0+}}$. (Strictly speaking our input in Section 3.6, the shortened G-datum, also requires the choice of a depth-zero type (K^0, ρ^0) (for which x satisfies the desired genericity condition),

[3]In [DF] the group $K_{x,+}$ is denoted by $K^+_{\iota,x}$. We use + as subscript here to have it better match the notation in Section 3.6.

however, the construction of $K_{x,+}$ and $\check{\phi}^+_{\iota,x}$ does not depend on the choice of depth-zero datum and works for arbitrary $x \in \mathcal{B}(G_\iota, F)$.) The character $\check{\phi}^+_{\iota,x}$ depends only on $\check{\varphi}|_{G_{\iota,x,0+}}$, and hence only on ϕ and not on the choice of extension φ of ϕ, which justifies the notation $\check{\phi}^+_{\iota,x}$.

We fix[4] an R-valued Haar measure on $G(F)$ and we write $\mathcal{H}_R(G)$ for the full Hecke algebra of $G(F)$, which consists of locally constant, compactly supported R-valued functions on $G(F)$ with the product given by convolution. We attach to $(K_{x,+}, \check{\phi}^+_{\iota,x})$ the idempotent $e_{\iota,x} \in \mathcal{H}_R(G)$ given by

$$e_{\iota,x} := \frac{1}{\mathrm{meas}(K_{x,+})} \begin{cases} \check{\phi}^+_{\iota,x}(g)^{-1} & \text{if } g \in K_{x,+} \\ 0 & \text{if } g \notin K_{x,+}. \end{cases}$$

Then $e_{\iota,x}$ acts on any smooth R-representation $\pi : G(F) \to \mathrm{Aut}_R(V_\pi)$ via

$$e_{\iota,x}(v) = \int_{G(F)} e_{\iota,x}(g)\pi(g)(v)dg = \frac{1}{\mathrm{meas}(K_{x,+})} \int_{K_{x,+}} \check{\phi}^+_{\iota,x}(g)^{-1}\pi(g)(v)dg \quad \text{for } v \in V_\pi,$$

and hence $e_{\iota,x}V_\pi$ is the subspace of V_π consisting of the $(K_{x,+}, \check{\phi}^+_{\iota,x})$-isotypic part of V_π, i.e., all those vectors on which $K_{x,+}$ acts by multiplication by $\check{\phi}^+_{\iota,x}$.

Step 3. In Step 2, we constructed an idempotent attached to ϕ, ι, x for $x \in \mathcal{B}(G_\iota, F)$, in the third step we use this construction to provide an idempotent for every point $x \in \mathcal{B}(G, F)$ that only depends on the conjugacy classes of ϕ and ι. To do so, let $x \in \mathcal{B}(G, F)$. It turns out [DF] that if $\iota_1, \iota_2 \in [\iota]$, where we recall that $[\iota]$ denotes the $G(F)$-conjugacy class of ι, and if $x \in \mathcal{B}(G_{\iota_1}) \cap \mathcal{B}(G_{\iota_2})$, then either $e_{\iota_1,x} = e_{\iota_2,x}$ or $e_{\iota_1,x}e_{\iota_2,x} = 0$. We write $i_1 \sim_x i_2$ if and only if $e_{\iota_1,x} = e_{\iota_2,x}$ and define the idempotent

$$e_x := e_{[\iota],x} := \sum_{\iota' \in [\iota] \text{ s.th. } x \in \mathcal{B}(G_{\iota'}, F) \;/\sim_x} e_{\iota',x}.$$

Note that the sum might be empty, in which case $e_{[\iota],x} = 0$.

Step 4. We can now define the subcategory $\mathrm{Rep}_R(G)^{[\phi,\iota]}$ as the full subcategory of $\mathrm{Rep}_R(G)$ that contains the following objects

$$\mathrm{Rep}_R(G)^{[\phi,\iota]} = \{V \in \mathrm{Rep}_R(G) \,|\, V = \sum_{x \in \mathcal{B}(G,F)} e_x V\}.$$

The basic example is the case in which ϕ is trivial. Then $\mathcal{I}_\phi$ consists of a single $G(F)$-conjugacy class, which we denote by $[\iota_0]$, we have $G_{\iota_0} = G$, $K_{x,+} = G_{x,0+}$, $\check{\phi}^+_{\iota_0,x} = \mathrm{triv}$, $e_x V = V^{G(F)_{x,0+}}$, and hence

$$\mathrm{Rep}_R(G)^{[\mathrm{triv},\iota_0]} = \{V \in \mathrm{Rep}_R(G) \,|\, V = \sum_{x \in \mathcal{B}(G,F)} V^{G_{x,0+}}\} = \mathrm{Rep}_R(G)_{\text{depth-zero}}.$$

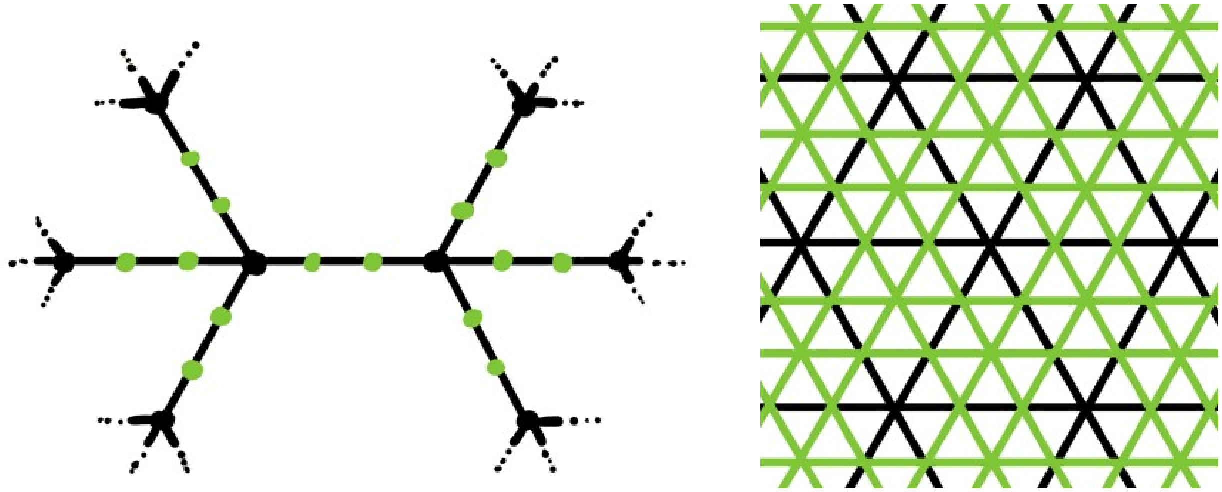
4.4 Decomposition of the category and reduction to depth-zero. Having introduced all the required notation, we can now restate Theorem 4.1 with a precise indexing set.

THEOREM 4.2 (Dat–Fintzen [DF]). *We recall that we assume that p does not divide the order of the absolute Weyl group of G, and R is a $\mathbb{Z}[\mu_{4p^\infty}, 1/\sqrt{p}]$-algebra. Then we have equivalences of categories*

$$\mathrm{Rep}_R(G) \simeq \prod_{[\phi] \in \Phi(P_F, G), [\iota] \in \mathcal{I}_\phi} \mathrm{Rep}_R(G)^{[\phi,\iota]} \quad \text{and} \quad \mathrm{Rep}_R(G)^{[\phi,\iota]} \simeq \mathrm{Rep}_R(G_\iota)_{\text{depth-zero}}.$$

The proof of the first equivalence relies on the idempotents above satisfying nice compatibility properties and on Theorem 3.11.

The proof of the reduction-to-depth-zero isomorphism $\mathrm{Rep}_R(G)^{[\phi,\iota]} \simeq \mathrm{Rep}_R(G_\iota)_{\text{depth-zero}}$ uses the notion of coefficient systems that we introduce in the next subsection before sketching the strategy of the proof in Section 4.6.



(a) (b)

Figure 4.1: Subdivision (in light green) for $e = 3$ of (a) a subset of the Bruhat–Tits building of $\mathrm{SL}_2(\mathbb{Q}_3)$ (in black) (b) a subset of an apartment of the Bruhat–Tits building of $\mathrm{SL}_3(F)$ (in black)

4.5 Equivariant coefficient systems on the Bruhat–Tits building. We fix $\phi \in [\phi] \in \Phi(P_F, G), \iota \in [\iota] \in \mathcal{I}_\phi$ for the remainder of the article.

We let e be a sufficiently divisible integer as in [DF, 2.7.4] and subdivide the polysimplicial structure of the Bruhat–Tits building $\mathcal{B}(G, F)$ by adding in between each two neighboring parallel hyperplanes e-1 additional equally spaced hyperplanes. See Figure 4.1 for an example. We denote this subdivided polysimplicial complex by $\mathcal{B}_{\bullet/e}(G)$ and refer to its facets as e-facets. Since we choose the integer e sufficiently divisible, if x and y are contained in the same e-facet $\mathcal{F} \in \mathcal{B}_{\bullet/e}(G)$, then $e_x = e_y$, and hence we may write $e_\mathcal{F} := e_x = e_y$.

If $V \in \mathrm{Rep}_R(G)^{[\phi,\iota]}$, then by definition $V = \sum_{x \in \mathcal{B}(G,F)} e_x V = \sum_{\mathcal{F} \in \mathcal{B}_{\bullet/e}(G)} e_\mathcal{F} V$. Hence all the information about V is contained in the collection $\{e_\mathcal{F} V\}_{\mathcal{F} \in \mathcal{B}_{\bullet/e}(G)}$ together with the action of $G(F)$ on this collection and information on how the different subspaces $e_\mathcal{F} V$ of V interact. This idea is made more precise with the notion of a $G(F)$-equivariant coefficient system.

Coefficient systems on the Bruhat–Tits building had already been introduced in the 1990s by Schneider–Stuhler [SS97] to study complex representations of p-adic groups and were extended to a framework with more general idempotents by Meyer–Solleveld [MS10]. Dat [Dat18a] used a generalization of them to decompose the category of depth-zero $\bar{\mathbb{Z}}_\ell$-representations of GL_n into blocks and show their equivalence with a principal block of a product of general linear groups (a feature special to GL_n), based also on work on coefficients systems for GL_n of his student Wang [Wan17]. Lanard then used similar techniques to study depth-zero representations of more general p-adic groups [Lan18, Lan21b, Lan21a, Lan23].

We introduce the notion of a $G(F)$-equivariant coefficient system following [DF, §3.1], which in turns follows [Dat18a, §3.1.1]. We denote by $[\mathcal{B}_{\bullet/e}(G)/G]$ the category whose objects are the set of e-facets $\mathcal{B}_{\bullet/e}(G)$ and whose set of morphisms is given by $\mathrm{Hom}(\mathcal{F}, \mathcal{F}') := \{g \in G(F) \,|\, \overline{g\mathcal{F}} \supseteq \mathcal{F}'\}$, where $\overline{g\mathcal{F}}$ denotes the closure of $g\mathcal{F}$, and with composition of morphisms given by multiplication in $G(F)$.

Definition 4.3. *A* $G(F)$-equivariant coefficient system of R-modules *is a functor* $\mathcal{V}$ *from* $[\mathcal{B}_{\bullet/e}(G)/G]$ *to the category of* R*-modules.*

Explicitly this means that a $G(F)$-equivariant coefficient system of R-modules is given by a collection of R-modules

$$\{\mathcal{V}_\mathcal{F}\}_{\mathcal{F} \in \mathcal{B}_{\bullet/e}(G)},$$

[4] To avoid the choice of a Haar measure one may work with distributions instead of functions.

with a collection of R-module homomorphisms

$$\{\beta_{\mathcal{F},\mathcal{F}'} : \mathcal{V}_{\mathcal{F}} \to \mathcal{V}_{\mathcal{F}'}\}_{\mathcal{F},\mathcal{F}'\in\mathcal{B}_{\bullet/e}(G) \text{ s.t. } \overline{\mathcal{F}}\supseteq\mathcal{F}'},$$

corresponding to $g = 1 \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$, and a collection of isomorphisms

$$\{g_{\mathcal{F}} : \mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} \mathcal{V}_{g\mathcal{F}}\}_{\mathcal{F}\in\mathcal{B}_{\bullet/e}(G), g\in G(F)},$$

subject to appropriate transitivity relations and compatibility conditions. We call such a $G(F)$-equivariant coefficient system $\mathcal{V}$ *smooth* if the action of the stabilizer $G_{\mathcal{F}}$ of $\mathcal{F}$ in $G(F)$ on $\mathcal{V}_{\mathcal{F}}$ given by $g_{\mathcal{F}}$ for $g \in G_{\mathcal{F}}$ is smooth. If $\mathcal{F}$ and $\mathcal{F}'$ are two e-facets in $\mathcal{B}_{\bullet/e}(G)$ with $\overline{\mathcal{F}} \supseteq \mathcal{F}'$, and $x \in \mathcal{F}$, then $K_x^+ \subset G_{\mathcal{F}'}$. Hence the idempotent e_x acts on R-representations of $G_{\mathcal{F}'}$ and we can make the following definition.

Definition 4.4. *We define* $\mathrm{Coef}_R^{[\phi,\iota]}(\mathcal{B}_{\bullet/e}(G)/G)$ *to be the full subcategory of* $G(F)$*-equivariant coefficients systems of* R*-modules whose objects are the smooth coefficient systems* $\mathcal{V}$ *for which for every pairs* $\mathcal{F}, \mathcal{F}' \in \mathcal{B}_{\bullet/e}(G)$ *with* $\overline{\mathcal{F}} \supseteq \mathcal{F}'$ *the morphism* $\beta_{\mathcal{F},\mathcal{F}'} : \mathcal{V}_{\mathcal{F}} \to \mathcal{V}_{\mathcal{F}'}$ *factors through* $e_{\mathcal{F}}V_{\mathcal{F}'}$ *and induced an isomorphism* $\mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} e_{\mathcal{F}}\mathcal{V}_{\mathcal{F}'}$.

As was our motivation for the definition, we obtain the following equivalence of categories, which is proven in [DF] by adjusting the arguments of Lanard [Lan21a] to our slightly more general setting.

Proposition 4.5. *We have an equivalence of categories* $\mathrm{Rep}_R(G)^{[\phi,\iota]} \simeq \mathrm{Coef}_R^{[\phi,\iota]}(\mathcal{B}_{\bullet/e}(G)/G)$.

The equivalence is obtained by associating to $V \in \mathrm{Rep}_R(G)^{[\phi,\iota]}$ the coefficient system $\{e_{\mathcal{F}}V\}_{\mathcal{F}\in\mathcal{B}_{\bullet/e}(G)}$ with the maps $\beta_{\mathcal{F},\mathcal{F}'}$ being inclusions and the maps $g_{\mathcal{F}}$ arising from the action of g on V. The quasi-inverse is obtained by taking an appropriate colimit.

Similarly, we can define a coefficient system that provides an equivalence with $\mathrm{Rep}_R(G_\iota)_{\text{depth-zero}} = \mathrm{Rep}_R(G_\iota)^{[\mathrm{triv},\iota_0]}$. In order to be able to relate it to the above coefficient system, we will use the polysimplicial structure of the subdivided Bruhat–Tits building of G intersected with the Bruhat–Tits building of G_ι. We denote the resulting polysimplicial complex by $\mathcal{B}_{\bullet/e}(G_\iota)$ and write $G_{\iota,\mathcal{F},0+} := G_{\iota,x,0+}$ for $x \in \mathcal{F}$. The subdivision is chosen sufficiently fine so that this group does not depend on x.

Definition 4.6. *We define* $\mathrm{Coef}_R^{[\mathrm{triv},\iota_0]}(\mathcal{B}_{\bullet/e}(G_\iota)/G_\iota)$ *to be the full subcategory of* $G_\iota(F)$*-equivariant coefficient systems of* R*-modules on* $\mathcal{B}_{\bullet/e}(G_\iota)$ *whose objects are the smooth coefficient systems* $\mathcal{V}$ *for which for every pair* $\mathcal{F}, \mathcal{F}' \in \mathcal{B}_{\bullet/e}(G_\iota)$ *with* $\overline{\mathcal{F}} \supseteq \mathcal{F}'$ *the morphism* $\beta_{\mathcal{V},\mathcal{F},\mathcal{F}'} : \mathcal{V}_{\mathcal{F}} \to \mathcal{V}_{\mathcal{F}'}$ *factors through* $V_{\mathcal{F}'}^{G_{\iota,\mathcal{F},0+}}$ *and induces an isomorphism* $\mathcal{V}_{\mathcal{F}} \xrightarrow{\sim} \mathcal{V}_{\mathcal{F}'}^{G_{\iota,\mathcal{F},0+}}$.

Like above, associating to $V \in \mathrm{Rep}_R(G_\iota)^{[\mathrm{triv},\iota_0]}$ the coefficient system $\{V^{G_{\iota,\mathcal{F},0+}}\}_{\mathcal{F}\in\mathcal{B}_{\bullet/e}(G_\iota)}$ with the maps $\beta_{\mathcal{F},\mathcal{F}'}$ being inclusions and the maps $g_{\mathcal{F}}$ arising from the action of g on V induces the following equivalence of categories.

Proposition 4.7. *We have an equivalence of categories* $\mathrm{Rep}_R(G_\iota)_{\text{depth-zero}} = \mathrm{Rep}_R(G_\iota)^{[\mathrm{triv},\iota_0]} \simeq \mathrm{Coef}_R^{[\mathrm{triv},\iota_0]}(\mathcal{B}_{\bullet/e}(G_\iota)/G_\iota)$.

4.6 Sketch of the proof of the reduction to depth zero. In this section we sketch the strategy of the proof in [DF] of the second equivalence in Theorem 4.2. Combining Proposition 4.5 and Proposition 4.7, it suffices to show that

$$\mathrm{Coef}_R^{[\phi,\iota]}(\mathcal{B}_{\bullet/e}(G)/G) \simeq \mathrm{Coef}_R^{[\mathrm{triv},\iota_0]}(\mathcal{B}_{\bullet/e}(G_\iota)/G_\iota). \tag{4.1}$$

For technical details in the proof that require the reduced Bruhat–Tits buildings of G_ι and of G to have the same dimension, we first reduce proving the second equivalence in Theorem 4.2 to the case that the center $Z(G_\iota)$ of G_ι modulo the center $Z(G)$ of G is compact using suitable results about compatibility with parabolic induction. So we assume from now on that $Z(G_\iota)/Z(G)$ is compact and sketch the proof of (4.1).

Let $\mathcal{F} \in \mathcal{B}_{\bullet/e}(G_\iota)$ and let $x \in \mathcal{F}$. Above we attached to x and $\check{\varphi}$ a compact, open subgroup $K_{x,+}$ with a character $\check{\phi}_x^+$ using the same construction as the one of types in Section 3.6. Since $K_{x,+}$, $\check{\phi}_x^+$ and $e_{\iota,x}$ only depend on $\mathcal{F}$, and not on the choice of x, we may also denote them by $K_{\mathcal{F},+}$, $\check{\phi}_{\mathcal{F}}^+$ and $e_{\iota,\mathcal{F}}$. Using the same construction as for the construction of types in Section 3.6, we can also construct a larger compact[5], open subgroup $K_{\mathcal{F}}$ (denoted

[5]We caution the expert that the analogous notation in [DF, §3] denotes the corresponding compact-mod-center, open subgroup, and that relatedly in [DF, §3] we work with the reduced Bruhat–Tits building, rather than the extended one that we are using here to stay in line with the construction of types.

by K in Section 3.6) and an irreducible representation $\kappa_{\mathcal{F}}$ (denoted κ in Section 3.6) of $K_{\mathcal{F}}$ whose restriction to $K_{\mathcal{F},+}$ is $\check{\phi}^+_{\mathcal{F}}$-isotypic. Using the theory of Weil–Heisenberg representations, which is extended to R-representations in [DF], we obtain an equivalence of categories between

the category of R-representations of $K_{\mathcal{F}}$ whose restriction to $K_{\mathcal{F},+}$ is $\check{\phi}^+_{\mathcal{F}}$-isotypic

and

the category of depth-zero representations of $G_{\iota,\mathcal{F}}$, the stabilizer of $\mathcal{F}$ in $G_\iota(F)$,

i.e., in symbols (representing the categories by the objects of the full subcategories that we are considering)

$$\{V \in \mathrm{Rep}_R(K_{\mathcal{F}}) \,|\, V = e_{\iota,\mathcal{F}}V\} \simeq \{V \in \mathrm{Rep}_R(G_{\iota,\mathcal{F}}) \,|\, V = V^{G_{\iota,\mathcal{F},0+}}\}. \tag{4.2}$$

The equivalence of categories is given by sending an object V_ι from the right hand side to the object $\kappa_{\mathcal{F}} \otimes_R V_\iota$ on the left hand side, and an object V from the left hand side to the object $\mathrm{Hom}_{K_{\mathcal{F},+}}(\kappa_{\mathcal{F}}, V)$ on the right hand side.

The strategy of proving the equivalence (4.1) is to "upgrade" the "local" equivalence (4.2) to an equivalence of coefficient systems. This is achieved in [DF] through the construction of an auxiliary equivariant smooth coefficient system on $\mathcal{B}_{\bullet/e}(G_\iota)$ that attaches to each $\mathcal{F} \in \mathcal{B}_{\bullet/e}(G_\iota)$ the above R-representation $\kappa_{\mathcal{F}}$ of $K_{\mathcal{F}}$ and that satisfies additional compatibility properties for adjacent facets, see [DF, Theorem 3.1.13] for details. Proving that such a coefficient system exists with all the required structural properties is a key result of [DF]. One crucial input is that we work with twisted Weil–Heisenberg representations that incooperates the quadratic character arising from the work of Fintzen–Kaletha–Spice [FKS23]. We refer the interested reader to [DF, §3.3-3.5] for the details. Using this auxiliary coefficient system to obtain the equivalence (4.1) requires some care as the auxiliary coefficient system and $\mathrm{Coef}_R^{[\mathrm{triv},\iota_0]}(\mathcal{B}_{\bullet/e}(G_\iota)/G_\iota)$ are only coefficient systems on $\mathcal{B}_{\bullet/e}(G_\iota)$ while $\mathrm{Coef}_R^{[\phi,\iota]}(\mathcal{B}_{\bullet/e}(G)/G)$ is a coefficient system on all of $\mathcal{B}_{\bullet/e}(G)$. The interested reader can find the details in [DF, §3.2].

Acknowledgments. The author thanks Jeffrey Adler, Jean-François Dat and Kazuma Ohara for providing very quick feedback on an earlier draft of this paper and spotting a few typos.

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