A DEPTH-ZERO PRINCIPAL-SERIES BLOCK WHOSE HECKE ALGEBRA HAS A NON-TRIVIAL TWO-COCYCLE

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Abstract. Recently the authors have shown that every Hecke algebra associated to a type constructed by Kim and Yu is isomorphic to a Hecke algebra for a depth-zero type. An example in the literature has been suggested as a counterexample to this result. We show that the example is not a counterexample, and exhibit some of its interesting properties, e.g., we show that a principal series, depth-zero type can have a Hecke algebra with non-trivial two-cocyle, a phenomenon that many did not expect could occur.

1. Introduction

Let G denote a connected reductive group over a non-archimedean local field F. The category $\mathrm{Rep}(G(F))$ of smooth, complex representations of $G(F)$ is a direct product of full subcategories called "Bernstein blocks":

$$\mathrm{Rep}(G(F)) = \prod_{\mathfrak{s}\in\mathfrak{I}(G)} \mathrm{Rep}^{\mathfrak{s}}(G(F)).$$

Each of the blocks $\mathrm{Rep}^{\mathfrak{s}}(G(F))$ is equivalent to the category of unital right modules over an algebra $\mathcal{H}^{\mathfrak{s}}$. Suppose that the category $\mathrm{Rep}^{\mathfrak{s}}(G(F))$ has an associated *type*, as defined by Bushnell and Kutzko [BK98], i.e., a compact open subgroup K of $G(F)$ and an irreducible smooth representation ρ of K such that a representation $\pi \in \mathrm{Rep}(G(F))$ belongs to $\mathrm{Rep}^{\mathfrak{s}}(G(F))$ if and only if every irreducible subquotient of π contains ρ upon restriction to K. Then we can replace the algebra $\mathcal{H}^{\mathfrak{s}}$ by the Hecke algebra $\mathcal{H}(G(F),(K,\rho))$ of all compactly supported, $\mathrm{End}_{\mathbb{C}}(\rho)$-valued functions on $G(F)$ that transform on the left and right according to ρ. That is, $\mathrm{Rep}^{\mathfrak{s}}(G(F))$ is equivalent to the category of modules over $\mathcal{H}(G(F),(K,\rho))$.

One of the present authors [Fin21] has shown that, provided that G splits over a tamely ramified extension of F and the residual characteristic p of F is not too small, the construction of Kim and Yu [KY17] provides types for every Bernstein block for $G(F)$. Thus, under this mild tameness assumption, one can in principle understand the category $\mathrm{Rep}(G(F))$ by understanding the structures of all of the Hecke algebras that arise from the types constructed by Kim and Yu.

In the "depth-zero" case, the compact group K contains a parahoric subgroup of $G(F)$, and the representation ρ is trivial on the pro-p radical of this parahoric. In the special case where K is a parahoric subgroup, Morris [Mor93, Theorem 7.12] has described the structures of these Hecke algebras. More generally, the authors [AFMO24a, Theorem 5.3.6] have described the structures of all depth-zero types.

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In the set-up of Kim and Yu ([KY17]), a type (K, ρ) for G of positive depth is constructed from a depth-zero type (K^0, ρ^0) for a subgroup G^0 of G, together with some additional data. The authors have recently shown [AFMO24b, Theorem 4.4.1] that the associated Hecke algebras $\mathcal{H}(G(F),(K,\rho))$ and $\mathcal{H}(G^0(F),(K^0,\rho^0))$ are isomorphic, after twisting the construction by Kim and Yu by a quadratic character arising from [FKS23], thus describing the structures of all Hecke algebras arising from Kim–Yu types.

In outline, from the pair (K, ρ), one constructs a group $W^\heartsuit$, and a normal, affine reflection subgroup W_{aff} of $W^\heartsuit$. Choosing a set S of generating reflections for W_{aff}, one constructs a parameter function $q\colon S \longrightarrow \mathbb{C}^\times$, thus obtaining an abstract Hecke algebra $\mathcal{H}(W_{\mathrm{aff}}, q)$. The choice of S gives rise to a complement Ω to W_{aff} in $W^\heartsuit$. A choice of a family $\mathcal{T}$ of intertwining operators gives rise to a 2-cocycle $\mu^{\mathcal{T}}\colon \Omega \times \Omega \longrightarrow \mathbb{C}^\times$. One then obtains an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{H}(G(F),(K,\rho)) \xrightarrow{\sim} \mathbb{C}[\Omega, \mu^{\mathcal{T}}] \ltimes \mathcal{H}(W_{\mathrm{aff}}, q).$$

That is, $\mathcal{H}(G(F),(K,\rho))$ is isomorphic to a semidirect product of our abstract Hecke algebra and the $\mu^{\mathcal{T}}$-twisted group algebra of Ω, where the structure of multiplication between these two factors is controlled by the conjugation action of Ω on W_{aff}.

In relation to the above discussion, Roche [Roc02, §4] and Goldberg–Roche [GR05, §11.8] each illustrate some unusual phenomena by presenting an example, that they attribute to Kutzko, of a Hecke algebra $\mathcal{H} := \mathcal{H}(G(F),(K,\rho))$ for a particular block of $G = \mathrm{SL}_8$. In this note, we discuss this example, determine an attached depth-zero pair (K^0, ρ^0) and describe the depth-zero algebra $\mathcal{H}^0 := \mathcal{H}(G^0(F),(K^0,\rho^0))$ that corresponds to it via [AFMO24b, Theorem 4.4.1] explicitly, as well as the closely related Hecke algebra $\mathcal{H}^{0,\circ} := \mathcal{H}(G^0(F),(K^{0,\circ},\rho^0))$, where we replace K^0 by the parahoric subgroup $K^{0,\circ}$ contained in it. We have several aims in doing so.

(a) First, a remark in [GR05] that $\mathcal{H}$ cannot be isomorphic to any of the intertwining algebras constructed by Morris [Mor93] let several mathematicians believe that $\mathcal{H}$ would provide a counterexample to [AFMO24b, Theorem 4.4.1]. Thus, we want to assure readers that this is not the case.
(b) Second, $\mathcal{H}^0$ provides an example of a depth-zero Hecke algebra where the associated affine reflection group is trivial, the group $\Omega(\rho_M)$ is nonabelian and infinite, and the cocycle $\mu^{\mathcal{T}}$ is non-trivial. In particular, $\mathcal{H}^0$ is an example of a Hecke algebra attached to a depth-zero, principal-series block of a quasi-split group that requires a non-trivial 2-cocycle, something that was long believed not to exist. We believe that this example might be useful for researchers in the future.

Notation. For a connected reductive group G and a reductive subgroup M of G, let $N_G(M)$, resp., $Z_G(M)$, denote the normalizer, resp., centralizer, of M in G.

For a finite field extension E/F and A a linear algebraic group or a Lie algebra thereof defined over E, we write $\mathrm{Res}_{E/F}(A)$ for the Weil restriction of A to F.

For a linear algebraic group G, we denote by $\mathrm{Lie}(G)$ the Lie algebra of G and by $\mathrm{Lie}^*(G)$ the dual of $\mathrm{Lie}(G)$. We also write $\mathrm{Lie}^*(G)^G$ for the subscheme of $\mathrm{Lie}^*(G)$ fixed by the coadjoint action of G on $\mathrm{Lie}^*(G)$. For a morphism $f\colon G \to H$ of algebraic groups, let

$$\mathrm{Lie}(f)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$$

denote the morphism between their Lie algebras induced by f.

Suppose that G is a connected reductive group defined over a non-archimedean local field F. We denote by $\mathcal{B}(G,F)$ the enlarged Bruhat–Tits building of G. For $x \in \mathcal{B}(G,F)$, let $G(F)_x$ denote the stabilizer of x in $G(F)$. For $r \in \mathbb{R}$ with $r \geq 0$, we also let $G(F)_{x,r}$, resp., $G(F)_{x,r+}$, be the Moy–Prasad filtration subgroup of $G(F)$ of depth r, resp., $r+$, associated to x (see [MP94, MP96]). We use the analogous notation for the Lie algebra $\mathrm{Lie}(G)$ and its dual $\mathrm{Lie}^*(G)$, where r is allowed to be any element of $\mathbb{R}$.

For a compact, open subgroup K of $G(F)$ and an irreducible smooth representation ρ of K, we denote by $\mathcal{H}(G(F),(K,\rho))$ the Hecke algebra attached to (K,ρ). We refer to [AFMO24a, Section 2.2] for the precise definition of $\mathcal{H}(G(F),(K,\rho))$.

Suppose that K is a subgroup of a group H and $h \in H$. We denote hKh^{-1} by ${}^hK$. If ρ is a representation of K, we write ${}^h\rho$ for the representation $x \mapsto \rho(h^{-1}xh)$ of ${}^hK$.

2. The example

In this section we introduce the example studied in this paper that we learned about from Roche [Roc02, §4] and Goldberg–Roche [GR05, §11.8], who attribute it to Kutzko. We present it, a type for the group SL_8, in the language of Kim and Yu's construction of types. Doing so then allows us to describe the associated depth-zero type for a smaller group G^0, seeing directly that it fits into our set-up.

Let F denote a non-archimedean local field with residue field $\mathfrak{f}$ of characteristic p (assumed odd) and order q. We fix a uniformizer ϖ_F of F and a square root $\sqrt{-1}$ of -1 in $\mathbb{C}^\times$. For any finite field extension E/F, we denote by $\mathcal{O}_E$ the ring of integers in E, by $\mathfrak{p}_E$ the prime ideal in $\mathcal{O}_E$, by $\mathrm{Tr}_{E/F}\colon E \to F$ the trace map, and by $N_{E/F}\colon E^\times \to F^\times$ the norm map. Let ord denote the discrete valuation on $F^\times$ with the value group $\mathbb{Z}$. For any finite extension E of F , we also write ord for the unique extension of this valuation to $E^\times$. We denote by $\mathrm{ord}_E^{\mathrm{norm}}\colon E^\times \twoheadrightarrow \mathbb{Z}$ the normalized valuation on $E^\times$.

Let ζ be a primitive $(q-1)$-st root of unity in F. Assume that 4 divides $q-1$. It follows that there exists a unique character $\eta\colon F^\times \to \mathbb{C}^\times$ that is trivial on ϖ_F and $1+\mathfrak{p}_F$ and satisfies $\eta(\zeta) = \sqrt{-1}$. Let E_2 be the splitting field of the polynomial $X^2 + \varpi_F$, and E_4 the splitting field of the polynomial $X^4 + \zeta\varpi_F$. (Note that the fields that we denote by E_2 and E_4 here are denoted by E_1 and E_2, respectively, in [Roc02].) Let ϖ_{E_2}, resp., ϖ_{E_4}, denote a uniformizer of E_2, resp., E_4, such that $\varpi_{E_2}^2 = -\varpi_F$, resp., $\varpi_{E_4}^4 = -\zeta\varpi_F$. We fix a generator σ_2 of the Galois group $\mathrm{Gal}(E_2/F)$ and a generator σ_4 of the Galois group $\mathrm{Gal}(E_4/F)$.

We define the following reductive groups over F:

$$\begin{aligned}
\widetilde{G} = \widetilde{G}^2 &= \mathrm{GL}_8, \\
\widetilde{G}^1 &= \mathrm{Res}_{E_2/F}(\mathrm{GL}_2) \times \mathrm{GL}_4, \\
\widetilde{G}^0 &= \mathrm{Res}_{E_2/F}(\mathrm{GL}_2) \times \mathrm{Res}_{E_4/F}(\mathrm{GL}_1), \\
\widetilde{T} = \widetilde{M}^0 &= \mathrm{Res}_{E_2/F}\left(\mathrm{GL}_1 \times \mathrm{GL}_1\right) \times \mathrm{Res}_{E_4/F}(\mathrm{GL}_1).
\end{aligned}$$

We identify $\mathrm{GL}_1 \times \mathrm{GL}_1$ with the diagonal torus of GL_2 by the map $(t_1, t_2) \mapsto \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right)$, thus obtaining an embedding $\widetilde{M}^0 \hookrightarrow \widetilde{G}^0$. Fix isomorphisms $F^{\oplus 2} \cong E_2$ and $F^{\oplus 4} \cong E_4$ of vector spaces over F, thus determining isomorphisms

$$F^{\oplus 8} \quad \cong \quad E_2^{\oplus 2} \oplus F^{\oplus 4} \quad \cong \quad E_2^{\oplus 2} \oplus E_4.$$

These choices determine embeddings of F-groups $\widetilde{G}^0 \hookrightarrow \widetilde{G}^1 \hookrightarrow \widetilde{G}^2$. Let us identify each group above with its images under these maps, so that they are all contained within $\widetilde{G} = \mathrm{GL}_8$.

The maximal split subtorus $A_{\widetilde{T}}$ of $\widetilde{T}$ is isomorphic to $\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1$. Note that $\widetilde{M}^0 = Z_{\widetilde{G}^0}(A_{\widetilde{T}})$. For $i = 1, 2$, let $\widetilde{M}^i = Z_{\widetilde{G}^i}(A_{\widetilde{T}})$, and write $\widetilde{M} = \widetilde{M}^2$.

Let $G = \mathrm{SL}_8$. For $X \in \{G^i, M^i, M, T \mid i = 0, 1, 2\}$, we let $X = \widetilde{X} \cap G$. We thus obtain a twisted Levi sequence $(G^0 \subset G^1 \subset G^2 = \mathrm{SL}_8)$, and a Levi subgroup $M^0 \subset G^0$. We denote by $\Phi(X, T)$ the absolute root system of X with respect to the maximal torus T. For $\alpha \in \Phi(X, T)$, we denote by $\alpha^\vee$ the corresponding (absolute) coroot.

We let $\widetilde{K}^0$ be the Iwahori subgroup of $\widetilde{G}^0(F)$ given by $\widetilde{K}^0 = I_2 \times I_4$, where I_2 denotes the Iwahori subgroup of $\mathrm{GL}_2(E_2) = \left(\mathrm{Res}_{E_2/F}(\mathrm{GL}_2)\right)(F)$ defined by

$$I_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(E_2) \;\middle|\; a, d \in \mathcal{O}_{E_2}^\times, b \in \mathcal{O}_{E_2}, c \in \mathfrak{p}_{E_2} \right\},$$

and I_4 denotes the Iwahori subgroup of $\left(\mathrm{Res}_{E_4/F}(\mathrm{GL}_1)\right)(F) = E_4^\times$, i.e., $I_4 = \mathcal{O}_{E_4}^\times$.

We choose $x_0 \in \mathcal{B}(M^0, F)$ and fix a commutative diagram $\{\iota\}$

$$\begin{array}{ccccc}
\mathcal{B}(G^0, F) & \hookrightarrow & \mathcal{B}(G^1, F) & \hookrightarrow & \mathcal{B}(G^2, F) \\
\uparrow & \circlearrowleft & \uparrow & \circlearrowleft & \uparrow \\
\mathcal{B}(M^0, F) & \hookrightarrow & \mathcal{B}(M^1, F) & \hookrightarrow & \mathcal{B}(M^2, F)
\end{array}$$

of admissible embeddings of buildings that is $(0, \frac{1}{8}, \frac{1}{4})$-generic relative to x_0 in the sense of [KY17, 3.5 Definition] such that $G^0(F)_{x_0} = \widetilde{K}^0 \cap G^0(F)$. Here and from now on, we identify a point in $\mathcal{B}(M^0, F)$ with its images via the embeddings $\{\iota\}$. Then we have

$$G^0(F)_{x_0,0} = \left\{(g_2, g_4) \in (I_2 \times I_4) \cap G^0(F) \mid (\det(g_2) \bmod \mathfrak{p}_{E_2}) \cdot (g_4 \bmod \mathfrak{p}_{E_4})^2 = 1\right\},$$

where we regard $(\det(g_2) \bmod \mathfrak{p}_{E_2})$ and $(g_4 \bmod \mathfrak{p}_{E_4})$ as elements of $\mathfrak{f}^\times$. Let K^0 be either $G^0(F)_{x_0}$ or $G^0(F)_{x_0,0}$. We also define $\widetilde{K}_{M^0} = \widetilde{K}^0 \cap \widetilde{M}^0(F)$ and $K_{M^0} = K^0 \cap M^0(F)$. Thus, we have $\widetilde{K}_{M^0} = \mathcal{O}_{E_2}^\times \times \mathcal{O}_{E_2}^\times \times \mathcal{O}_{E_4}^\times$ and $K_{M^0} = M^0(F)_{x_0}$ or $M^0(F)_{x_0,0}$ according as $K^0 = G^0(F)_{x_0}$ or $G^0(F)_{x_0,0}$. More precisely,
(2.1)

$$K_{M^0} = \begin{cases} \left\{(x, y, z) \in \mathcal{O}_{E_2}^\times \times \mathcal{O}_{E_2}^\times \times \mathcal{O}_{E_4}^\times \mid N_{E_2/F}(xy) N_{E_4/F}(z) = 1\right\} \\ \hfill \text{if } K^0 = G^0(F)_{x_0}, \\ \left\{(x, y, z) \in \mathcal{O}_{E_2}^\times \times \mathcal{O}_{E_2}^\times \times \mathcal{O}_{E_4}^\times \;\middle|\; \begin{array}{l} N_{E_2/F}(xy)\, N_{E_4/F}(z) = 1, \\ (xy \bmod \mathfrak{p}_{E_2}) \cdot (z \bmod \mathfrak{p}_{E_4})^2 = 1 \end{array} \right\} \\ \hfill \text{if } K^0 = G^0(F)_{x_0,0}. \end{cases}$$

We observe that

$$K^0 = K_{M^0} \cdot G^0(F)_{x_0,0}. \tag{2.2}$$

Since the embedding $\iota\colon \mathcal{B}(M^0, F) \hookrightarrow \mathcal{B}(G^0, F)$ is 0-generic relative to x_0, the inclusion $M^0(F)_{x_0,0} \subset G^0(F)_{x_0,0}$ induces an isomorphism

$$M^0(F)_{x_0,0}/M^0(F)_{x_0,0+} \xrightarrow{\sim} G^0(F)_{x_0,0}/G^0(F)_{x_0,0+}.$$

Combining this with (2.2), we also have

$$K_{M^0}/M^0(F)_{x_0,0+} \xrightarrow{\sim} K^0/G^0(F)_{x_0,0+}. \tag{2.3}$$

We define the character $\widetilde{\rho}_{M^0}$ of $\widetilde{K}_{M^0}$ by $\widetilde{\rho}_{M^0} = 1 \boxtimes \left(\eta \circ N_{E_2/F}\right) \boxtimes 1$, and write $\rho_{M^0} = \widetilde{\rho}_{M^0}|_{K_{M^0}}$. We define the character ρ^0 of K^0 as the composition of the surjection $K^0 \twoheadrightarrow K^0/G^0(F)_{x_0,0+}$, the inverse of the isomorphism in (2.3) and the character ρ_{M^0}. More precisely, ρ^0 is the restriction to K^0 of the character $\eta_2 \boxtimes 1$ of the group $\widetilde{K}^0$, where η_2 denotes the character of I_2 defined by

$$\eta_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\eta \circ N_{E_2/F}\right)(d).$$

Let $E = E_2$ or E_4. We fix an additive character $\Psi\colon F \to \mathbb{C}^\times$ that is trivial on $\mathfrak{p}_F$ and non-trivial on $\mathcal{O}_F$. We define a character ϕ_E of $1 + \mathfrak{p}_E$ by $\phi_E(1 + x) = \Psi(\mathrm{Tr}_{E/F}(\varpi_E^{-1}x))$ for $x \in \mathfrak{p}_E$. We fix an extension of ϕ_E to $E^\times$ and use the same notation ϕ_E for it. We also define the character $\phi_{\mathrm{GL}_2(E_2)}$ of $\mathrm{GL}_2(E_2)$ by $\phi_{\mathrm{GL}_2(E_2)}(g) = \phi_{E_2}(\det(g))$. We define the character $\widetilde{\phi}_0$ of $\widetilde{G}^0(F) = \mathrm{GL}_2(E_2) \times E_4^\times$ by $\widetilde{\phi}_0 = 1 \boxtimes \phi_{E_4}$, and define the character $\widetilde{\phi}_1$ of $\widetilde{G}^1(F) = \mathrm{GL}_2(E_2) \times \mathrm{GL}_4(F)$ by $\widetilde{\phi}_1 = \phi_{\mathrm{GL}_2(E_2)} \boxtimes 1$. We write $\phi_0 = \widetilde{\phi}_0|_{G^0(F)}$ and $\phi_1 = \widetilde{\phi}_1|_{G^1(F)}$.

Lemma 2.1. *The character ϕ_0 is (G^1, G^0)-generic of depth $\frac{1}{4}$ relative to the point x_0, and the character ϕ_1 is (G^2, G^1)-generic of depth $\frac{1}{2}$ relative to the point x_0 in the sense of* [Fin, Definition 3.5.2].

Proof. By construction, ϕ_0 is trivial on $G^0(F)_{x_0,(1/4)+}$, and ϕ_1 is trivial on $G^1(F)_{x_0,(1/2)+}$. We define $\widetilde{X}_0^* \in \mathrm{Lie}^*(\widetilde{G}^0)^{\widetilde{G}^0}(F)$ and $\widetilde{X}_1^* \in \mathrm{Lie}^*(\widetilde{G}^1)^{\widetilde{G}^1}(F)$ as follows. Let $E \in \{E_2, E_4\}$. We use the same notation

$$\mathrm{Tr}_{E/F}\colon\ \mathrm{Res}_{E/F}(\mathrm{Lie}(\mathrm{GL}_1)) \to \mathrm{Lie}(\mathrm{GL}_1)$$

for the usual trace morphism whose map on F-valued points is the trace map. Let

$$m(\varpi_E^{-1})\colon\ \mathrm{Lie}\left(\mathrm{Res}_{E/F}(\mathrm{GL}_1)\right) \to \mathrm{Lie}\left(\mathrm{Res}_{E/F}(\mathrm{GL}_1)\right)$$

denote the morphism induced by multiplication by $\varpi_E^{-1} \in E = \mathrm{Lie}\left(\mathrm{Res}_{E/F}(\mathrm{GL}_1)\right)(F)$. We define $\widetilde{X}_0^* \in \mathrm{Lie}^*(\widetilde{G}^0)^{\widetilde{G}^0}(F)$ as the composition of the projection map

$$\mathrm{Lie}(\widetilde{G}^0) \to \mathrm{Lie}\left(\mathrm{Res}_{E_4/F}(\mathrm{GL}_1)\right)$$

and

$$\mathrm{Tr}_{E_4/F} \circ m(\varpi_{E_4}^{-1})\colon\ \mathrm{Lie}\left(\mathrm{Res}_{E_4/F}(\mathrm{GL}_1)\right) \to \mathrm{Lie}(\mathrm{GL}_1).$$

To define $\widetilde{X}_1^*$, we let

$$\mathrm{Res}_{E_2/F}(\det)\colon\ \mathrm{Res}_{E_2/F}(\mathrm{GL}_2) \to \mathrm{Res}_{E_2/F}(\mathrm{GL}_1)$$

be the morphism of algebraic groups induced by the usual determinant map $\det\colon \mathrm{GL}_2 \to \mathrm{GL}_1$. Now, we define $\widetilde{X}_1^* \in \mathrm{Lie}^*(\widetilde{G}^1)^{\widetilde{G}^1}(F)$ as the composition of the projection map

$$\mathrm{Lie}(\widetilde{G}^1) \to \mathrm{Lie}\left(\mathrm{Res}_{E_2/F}(\mathrm{GL}_2)\right)$$

and

$$\mathrm{Tr}_{E_2/F} \circ m(\varpi_{E_2}^{-1}) \circ \mathrm{Lie}\left(\mathrm{Res}_{E_2/F}(\det)\right) : \mathrm{Lie}\left(\mathrm{Res}_{E_2/F}(\mathrm{GL}_2)\right) \to \mathrm{Lie}(\mathrm{GL}_1).$$

We define $X_0^* \in \mathrm{Lie}^*(G^0)^{G^0}(F)$ and $X_1^* \in \mathrm{Lie}^*(G^1)^{G^1}(F)$ as the restrictions of $\widetilde{X}_0^*$ and $\widetilde{X}_1^*$ to $\mathrm{Lie}(G^0)$ and $\mathrm{Lie}(G^1)$, respectively. Then the restriction of ϕ_0 to

$$G^0(F)_{x_0,1/4}/G^0(F)_{x_0,(1/4)+} \simeq \mathrm{Lie}(G^0)(F)_{x_0,1/4}/\,\mathrm{Lie}(G^0)(F)_{x_0,(1/4)+}$$

is given by $\Psi \circ X_0^*$, and the restriction of ϕ_1 to

$$G^1(F)_{x_0,1/2}/G^1(F)_{x_0,(1/2)+} \simeq \mathrm{Lie}(G^1)(F)_{x_0,1/2}/\,\mathrm{Lie}(G^1)(F)_{x_0,(1/2)+}$$

is given by $\Psi \circ X_1^*$.

We will prove that X_0^* is (G^1, G^0)-generic of depth $-1/4$, and X_1^* is (G^2, G^1)-generic of depth $-1/2$ in the sense of [Fin, Definition 3.5.2]. First, we will prove that X_0^* satisfies **(GE0)** and **(GE1)** in [Fin, Definition 3.5.2]. Let $\alpha \in \Phi(G^1, T) \smallsetminus \Phi(G^0, T)$. Then we have

$$X_0^*(\mathrm{Lie}(\alpha^\vee)(1)) = \sigma_4^i(\varpi_{E_4}^{-1}) - \sigma_4^j(\varpi_{E_4}^{-1})$$

for some $i, j \in \{0, 1, 2, 3\}$ with $i \neq j$. Since $E_4 = F[\varpi_{E_4}^{-1}]$, we obtain from [May20, Proposition 5.9] that

$$\mathrm{ord}\left(\sigma_4^i(\varpi_{E_4}^{-1}) - \sigma_4^j(\varpi_{E_4}^{-1})\right) = \mathrm{ord}(\varpi_{E_4}^{-1}) = -1/4.$$

Thus, the element X_0^* satisfies **(GE1)** in [Fin, Definition 3.5.2]. Moreover, since $(0, \varpi_{E_4}) \in \mathrm{Lie}(G^0)_{x_0,1/4}$ (where we view ϖ_{E_4} in $\mathrm{Lie}(\mathrm{Res}_{E_4/F}(\mathrm{GL}_1))(F)$ by identifying the latter with E_4 and note that $(0, \varpi_{E_4}) \in \mathrm{Lie}(G^0)(F)$ as ϖ_{E_4} has trace zero) and

$$\mathrm{ord}\left(X_0^*(0, \varpi_{E_4})\right) = \mathrm{ord}(4) = 0,$$

we have $X_0^* \notin \mathrm{Lie}^*(G^0)_{x_0,(-1/4)+}$. Since it can be checked from the definition that $X_0^* \in \mathrm{Lie}^*(G^0)_{x_0,-1/4}$, the element X_0^* also satisfies **(GE0)** in [Fin, Definition 3.5.2].

Next, we will prove that X_1^* satisfies **(GE0)** and **(GE1)** in [Fin, Definition 3.5.2]. Let $\alpha \in \Phi(G^2, T) \smallsetminus \Phi(G^1, T)$. Then, using that $\sigma_2(\varpi_{E_2}) = -\varpi_{E_2}$, we obtain

$$X_1^*(\mathrm{Lie}(\alpha^\vee)(1)) \in \{\pm\varpi_{E_2}^{-1}, \pm 2\varpi_{E_2}^{-1}\}.$$

Hence

$$\mathrm{ord}\left(X_1^*(\mathrm{Lie}(\alpha^\vee)(1))\right) = \mathrm{ord}(\varpi_{E_2}^{-1}) = -1/2,$$

and X_1^* satisfies **(GE1)** in [Fin, Definition 3.5.2]. Consider the element $\left(\begin{pmatrix} \varpi_{E_2} & 0 \\ 0 & 0 \end{pmatrix}, 0\right) \in \mathrm{Lie}^*(G^1)_{x_0,-1/2}$, then

$$\mathrm{ord}\left(X_1^*\left(\begin{pmatrix} \varpi_{E_2} & 0 \\ 0 & 0 \end{pmatrix}, 0\right)\right) = \mathrm{ord}(2) = 0,$$

thus $X_1^* \notin \mathrm{Lie}^*(G^1)_{x_0,-1/2+}$. Moreover, $X_1^* \in \mathrm{Lie}^*(G^1)_{x_0,-1/2}$, hence X_1^* satisfies **(GE0)** in [Fin, Definition 3.5.2].

Since the only possible torsion prime for the dual root datum of G^1 and G^2 is 2, and since $p \neq 2$, by [Yu01, Lemma 8.1] condition **(GE2)** is also satisfied. $\square$

As a consequence of the above lemma, the datum

$$\Sigma = \left((G^0 \subset G^1 \subset G^2, M^0), (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}), (x_0, \{\iota\}), (K_{M^0}, \rho_{M^0}), (\phi_0, \phi_1, 1)\right)$$

satisfies the properties of [KY17, (7.2)], in other words, it is a G-datum as in [AFMO24b, Definition 4.1.1]. Applying the construction of Kim and Yu in [KY17, 7.4] to Σ, we obtain a compact, open subgroup K of $G(F)$ and an irreducible smooth representation ρ of K.

Remark 2.2. In [AFMO24b], we twist the construction of Kim and Yu by a quadratic character $\epsilon_{x_0}^{\overrightarrow{G}}$ of $K^0/G^0(F)_{x_0,0+} \simeq K_{M^0}/M^0(F)_{x_0,0+}$ introduced in [FKS23], see [AFMO24b, §4.1] for details. In our case, we can compute $\epsilon_{x_0}^{\overrightarrow{G}}$ using [FKS23, Definition 3.1, Theorem 3.4] as follows:

$$\epsilon_{x_0}^{\overrightarrow{G}}\left((x, y, z) \bmod M^0(F)_{x_0,0+}\right) = \mathrm{sgn}_{\mathfrak{f}}(xy^{-1} \bmod \mathfrak{p}_{E_2}) = \mathrm{sgn}_{\mathfrak{f}}(xy \bmod \mathfrak{p}_{E_2})$$

for $(x, y, z) \in K_{M^0}$, where $\mathrm{sgn}_{\mathfrak{f}}$ denotes the unique non-trivial quadratic character of $\mathfrak{f}^\times = \mathcal{O}_{E_2}^\times/(1 + \mathfrak{p}_{E_2})$. Since $-1 \in (\mathfrak{f}^\times)^2$, the conditions in (2.1) imply that

$$xy \bmod \mathfrak{p}_{E_2} = \pm(z \bmod \mathfrak{p}_{E_4})^{-2} \in (\mathfrak{f}^\times)^2.$$

Thus, we obtain that $\epsilon_{x_0}^{\overrightarrow{G}}$ is trivial, and the twisted and non-twisted constructions agree in our case.

According to [AFMO24b, Theorem 4.4.1], we have an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{H}(G^0(F), (G^0(F)_{x_0}, \rho^0)) \xrightarrow{\sim} \mathcal{H}(G(F), (K, \rho)).$$

In the following section, we determine explicitly the structure of the Hecke algebras $\mathcal{H}(G^0(F), (G^0(F)_{x_0}, \rho^0))$ and $\mathcal{H}(G^0(F), (G^0(F)_{x_0,0}, \rho^0))$.

3. Structure of the depth-zero Hecke algebra

In this section, we will study the Hecke algebra $\mathcal{H}(G^0(F), (K^0, \rho^0))$ associated with the depth-zero type (K^0, ρ^0). We define the subgroup $N(\rho_{M^0})$ of the F-points of the normalizer $N_{G^0}(M^0)$ of M^0 in G^0 by

$$N(\rho_{M^0}) := \left\{n \in N_{G^0}(M^0)(F) \mid {}^nK_{M^0} = K_{M^0},\ {}^n\rho_{M^0} = \rho_{M^0}\right\}$$

and write $W(\rho_{M^0}) := N(\rho_{M^0})/K_{M^0}$. We write

$$I_{G^0(F)}(\rho^0) := \{g \in G^0(F) \mid \mathrm{Hom}_{K^0 \cap {}^gK^0}({}^g\rho^0, \rho^0) \neq \{0\}\}.$$

Then from [AFMO24a, Proposition 5.3.2 and Corollary 3.4.14], we have $N(\rho_{M^0}) \subset I_{G^0(F)}(\rho^0)$ and the inclusion induces a bijection

$$W(\rho_{M^0}) \simeq K^0 \backslash I_{G^0(F)}(\rho^0)/K^0.$$

In order to describe the groups $N(\rho_{M^0})$ and $W(\rho_{M^0})$ below, we define the element $\widetilde{s}$ of the group

$$G^0(F) = (\mathrm{GL}_2(E_2) \times \mathrm{GL}_1(E_4)) \cap \mathrm{SL}_8(F) \supset \mathrm{SL}_2(E_2) \times \mathrm{SL}_1(E_4)$$

by $\widetilde{s} = \left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right)$. Then we have $N_{G^0}(M^0)(F) = \{1, \widetilde{s}\} \ltimes M^0(F)$. We note that the element $\widetilde{s}$ normalizes the groups $\widetilde{K}_{M^0}$ and K_{M^0}.

Lemma 3.1. *The element $\widetilde{s}$ normalizes the character ρ_{M^0}.*

Proof. We have

$$\begin{aligned}
{}^{\widetilde{s}}\widetilde{\rho}_{M^0} &= {}^{\widetilde{s}}\Big(1 \boxtimes (\eta \circ N_{E_2/F}) \boxtimes 1\Big) \\
&= (\eta \circ N_{E_2/F}) \boxtimes 1 \boxtimes 1 \\
&= \Big(1 \boxtimes (\eta^{-1} \circ N_{E_2/F}) \boxtimes (\eta^{-1} \circ N_{E_4/F})\Big) \otimes (\eta \circ \det),
\end{aligned}$$

where det denotes the restriction of the determinant map $\mathrm{GL}_8(F) \to F^\times$ to the group $\widetilde{K}_{M^0}$. Since the group K_{M^0} is contained in the group $\mathrm{SL}_8(F)$, we have

$$ {}^{\widetilde{s}}\rho_{M^0} = \Big(1 \boxtimes (\eta^{-1} \circ N_{E_2/F}) \boxtimes (\eta^{-1} \circ N_{E_4/F})\Big)|_{K_{M^0}}. \tag{3.1}$$

We will prove that

$$\eta^2 \circ N_{E_2/F}|_{\mathcal{O}_{E_2}^\times} = 1 \tag{3.2}$$

and

$$\eta \circ N_{E_4/F}|_{\mathcal{O}_{E_4}^\times} = 1. \tag{3.3}$$

First, we will prove Equation (3.2). The definition of E_2 implies that we have

$$N_{E_2/F}(\mathcal{O}_{E_2}^\times) = (1 + \mathfrak{p}_F)\langle \zeta^2 \rangle.$$

Hence, Equation (3.2) follows from the definition of η. Similarly, we can prove Equation (3.3) by using the definition of η and the equation

$$N_{E_4/F}(\mathcal{O}_{E_4}^\times) = (1 + \mathfrak{p}_F)\langle \zeta^4 \rangle.$$

Combining equation (3.1) with Equations (3.2) and (3.3), we obtain that

$$\begin{aligned}
{}^{\widetilde{s}}\rho_{M^0} &= \Big(1 \boxtimes (\eta^{-1} \circ N_{E_2/F}) \boxtimes (\eta^{-1} \circ N_{E_4/F})\Big)|_{K_{M^0}} \\
&= \Big(1 \boxtimes (\eta \circ N_{E_2/F}) \boxtimes 1\Big)|_{K_{M^0}} \\
&= \widetilde{\rho}_{M^0}|_{K_{M^0}} = \rho_{M^0}.
\end{aligned}$$

□

Proposition 3.2. *We have*

$$N(\rho_{M^0}) = N_{G^0}(M^0)(F) = \{1, \widetilde{s}\} \ltimes M^0(F).$$

Proof. The claim $N(\rho_{M^0}) \subset N_{G^0}(M^0)(F)$ follows from the definition of $N(\rho_{M^0})$. We will prove the reverse inclusion. According to Lemma 3.1, we have $\widetilde{s} \in N(\rho_{M^0})$. Moreover, since M^0 is a torus, the conjugate action of $M^0(F)$ on K_{M^0} is trivial. Thus, we conclude that the group $M^0(F)$ normalizes the character ρ_{M^0}. □

To describe the structure of the group $W(\rho_{M^0})$, we define the elements $\widetilde{s}' \in N(\rho_{M^0})$ and $\widetilde{z} \in M^0(F)$ by $\widetilde{s}' = \left(\begin{pmatrix} 0 & \varpi_{E_2}^{-1} \\ -\varpi_{E_2} & 0 \end{pmatrix}, 1\right)$ and $\widetilde{z} = \left(\zeta\varpi_{E_2}, \varpi_{E_2}, \varpi_{E_4}^{-2}\right)$. Let s, s', and z be the images of $\widetilde{s}$, $\widetilde{s}'$, and $\widetilde{z}$ in $W(\rho_{M^0})$, respectively. When $K^0 = G^0(F)_{x_0,0}$, we also set $\widetilde{\epsilon}_{M^0} := (-1, 1, 1) \in M^0(F)_{x_0} \smallsetminus M^0(F)_{x_0,0}$ and let ϵ_{M^0} denote the image of $\widetilde{\epsilon}_{M^0}$ in $W(\rho_{M^0})$.

Corollary 3.3. *We have* $W(\rho_{M^0}) = \{1, s\} \ltimes (M^0(F)/K_{M^0})$.

Proof. This is immediate from Proposition 3.2. □

Proposition 3.4. *We have*

$$W(\rho_{M^0}) = \begin{cases} \langle s, s' \rangle \times \langle z \rangle & \simeq W_{\mathrm{aff}}(\widetilde{A}_1) \times \mathbb{Z} \qquad (K^0 = G^0(F)_{x_0}), \\ \langle s, s' \rangle \times \langle z, \epsilon_{M^0} \rangle & \simeq W_{\mathrm{aff}}(\widetilde{A}_1) \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad (K^0 = G^0(F)_{x_0,0}), \end{cases}$$

where $W_{\mathrm{aff}}(\widetilde{A}_1)$ denotes the affine Weyl group of the affine root system of type $\widetilde{A}_1$, i.e., the affine Weyl group with two simple reflections and no relation between them.

Proof. First, we consider the case where $K^0 = G^0(F)_{x_0}$. We define the isomorphism

$$H_{M^0} \colon \widetilde{M}^0(F)/\widetilde{K}_{M^0} \to \mathbb{Z}^3$$

by

$$(x, y, z)\widetilde{K}_{M^0} \mapsto (\mathrm{ord}_{E_2}^{\mathrm{norm}}(x), \mathrm{ord}_{E_2}^{\mathrm{norm}}(y), \mathrm{ord}_{E_4}^{\mathrm{norm}}(z)).$$

The definition of M^0 implies that an element $(n_1, n_2, n_3) \in \mathbb{Z}$ is contained in the image of the subgroup

$$M^0(F)/K_{M^0} \simeq M^0(F) \cdot \widetilde{K}_{M^0}/\widetilde{K}_{M^0}$$

of $\widetilde{M}^0(F)/\widetilde{K}_{M^0}$ if and only if

$$N_{E_2/F}(\varpi_{E_2}^{n_1+n_2}) \cdot N_{E_4/F}(\varpi_{E_4}^{n_3}) \in N_{E_2/F}(\mathcal{O}_{E_2}^\times) \cdot N_{E_4/F}(\mathcal{O}_{E_4}^\times). \tag{3.4}$$

The definition of E_2 and E_4 implies that (3.4) is equivalent to the condition that

$$\varpi_F^{n_1+n_2+n_3} \cdot \zeta^{n_3} \in (1 + \mathfrak{p}_F)\langle \zeta^2 \rangle.$$

Thus, we conclude that

$$\begin{aligned} H_{M^0}\left(M^0(F)/K_{M^0}\right) &= \{(n_1, n_2, n_3) \in \mathbb{Z} \mid n_1 + n_2 + n_3 = 0 \text{ and } 2 \mid n_3\} \\ &= \langle (1, 1, -2), (1, -1, 0) \rangle. \end{aligned}$$

Since $H_{M^0}(z) = (1, 1, -2)$, $H_{M^0}(ss') = (1, -1, 0)$, and H_{M^0} is an isomorphism, we have that

$$M^0(F)/K_{M^0} = \langle ss', z \rangle.$$

From Corollary 3.3, we thus obtain that

$$W(\rho_{M^0}) = \{1, s\} \ltimes \left(M^0(F)/K_{M^0}\right) = \langle s, s', z \rangle.$$

One can check that the subgroup $\langle s, s' \rangle$ of $W(\rho_{M^0})$ is isomorphic to the affine Weyl group of type $\widetilde{A_1}$, and that the element z has infinite order, commutes with the elements s and s', and no non-trivial power of z is contained in the span of s and s'.

Next, we consider the case where $K^0 = G^0(F)_{x_0,0}$. In this case, we have

$$M^0(F)/K_{M^0} = M^0(F)/M^0(F)_{x_0,0} = \left(M^0(F)/M^0(F)_{x_0}\right) \times \langle \epsilon_{M^0} \rangle.$$

Noting that ϵ_{M^0} commutes with s and s', the claim follows from the first case. $\square$

Lemma 3.5. *Let $n_s, n_z \in N(\rho_{M^0})$ denote lifts of s and z. Then, we have $[n_s, n_z] \in K_{M^0} \smallsetminus \ker \rho_{M^0}$. In particular, we have $n_s n_z \neq n_z n_s$.*

Proof. We write $n_s = \widetilde{s}k$ and $n_z = \widetilde{z}k'$ for some $k, k' \in K_{M^0}$. Then, we have

$$\begin{aligned}[n_s, n_z] &= [\widetilde{s}k, \widetilde{z}k'] \\ &= [\widetilde{s}k, \widetilde{z}] \cdot {}^{\widetilde{z}}[\widetilde{s}k, k'] \\ &= {}^{\widetilde{s}}[k, \widetilde{z}] \cdot [\widetilde{s}, \widetilde{z}] \cdot {}^{\widetilde{z}}[\widetilde{s}k, k'].\end{aligned}$$

Since $\widetilde{s}, \widetilde{z} \in N(\rho_{M^0})$ normalize ρ_{M^0}, we have

$${}^{\widetilde{s}}[k, \widetilde{z}], {}^{\widetilde{z}}[\widetilde{s}k, k'] \in \ker \rho_{M^0}.$$

On the other hand, we have

$$[\widetilde{s}, \widetilde{z}] = (\zeta^{-1}, \zeta, 1) \in K_{M^0} \smallsetminus \ker \rho_{M^0}$$

since

$$\rho_{M^0}((\zeta^{-1}, \zeta, 1)) = \left(\eta \circ N_{E_2/F}\right)(\zeta) = \eta(\zeta^2) = -1.$$

Thus, we conclude that $[n_s, n_z] \in K_{M^0} \smallsetminus \ker \rho_{M^0}$. □

Corollary 3.6. *The character ρ_{M^0} does not extend to the group $N(\rho_{M^0})$.*

Proof. Suppose that ρ_{M^0} extends to a character $\rho^{\dagger}_{M^0}$of $N(\rho_{M^0})$. Then, since $\widetilde{s}, \widetilde{z} \in N(\rho_{M^0})$, we have $[\widetilde{s}, \widetilde{z}] \in \ker \rho^{\dagger}_{M^0}$, which contradicts Lemma 3.5. □

Our decomposition of $W(\rho_{M^0})$ given in Proposition 3.4 gives rise to a length function on this group, the standard length function on extended affine Weyl groups. More precisely, we start with the length function ℓ_{prim} on $\langle s, s' \rangle$ with respect to the generators $\{s, s'\}$ of $\langle s, s' \rangle$. We extend ℓ_{prim} to $W(\rho_{M^0})$ by

$$\begin{cases} \ell_{\text{prim}}(wz^n) := \ell_{\text{prim}}(w) & \text{when } K^0 = G^0(F)_{x_0}, \\ \ell_{\text{prim}}(wz^n\epsilon^t_{M^0}) := \ell_{\text{prim}}(w) & \text{when } K^0 = G^0(F)_{x_0,0} \end{cases}$$

for $w \in \langle s, s' \rangle$, $n \in \mathbb{Z}$, and $t \in \{0, 1\}$.

Remark 3.7. Suppose that $K^0 = G^0(F)_{x_0,0}$. According to [Mor93, Proposition 5.2], we can take a lift $n_w \in N(\rho_{M^0})$ for each $w \in W(\rho_{M^0})$ such that if $\ell_{\text{prim}}(w_1w_2) = \ell_{\text{prim}}(w_1) + \ell_{\text{prim}}(w_2)$, then we have $n_{w_1w_2} = n_{w_1}n_{w_2}$. However, this statement is false in general, and Lemma 3.5 provides a counterexample. The failure of [Mor93, Proposition 5.2] does not affect Morris's main result [Mor93, Theorem 7.12] as his proof can easily be adapted to circumvent the use of such good coset representatives. Alternatively, the recent proof of the more general result [AFMO24a, Section 5] also does not rely on a choice of representatives. On the other hand, [Mor93, Remark 7.12(a)], which states that the the 2-cocycle $\mu^{\mathcal{T}}$ is trivial if the representation ρ^0 is a character, does depend on such representatives, and the example covered in this paper shows that [Mor93, Remark 7.12(a)] is not true in general (see Corollary 3.9 below).

Although Proposition 3.4 decomposes $W(\rho_{M^0})$ into a product of an affine Weyl group $\langle s, s' \rangle$ and a complement ($\langle z \rangle$ and $\langle z, \epsilon_{M^0} \rangle$ for $K^0 = G^0(F)_{x_0}$ and $K^0 = G^0(F)_{x_0,0}$, respectively) the decomposition of $W(\rho_{M^0})$ provided in [AFMO24a] (and also in [Mor93, 7.3]) is different, and comes from a different length function, where more elements have length zero, which is denoted by $\ell_{\mathcal{K}\text{-rel}}$ and defined in [AFMO24a, Definition 3.6.3]. The subgroup $\Omega(\rho_{M^0})$ of $W(\rho_{M^0})$ is defined by

$$\Omega(\rho_{M^0}) := \left\{ w \in W(\rho_{M^0}) \;\middle|\; \ell_{\mathcal{K}\text{-rel}}(w) = 0 \right\}.$$

Proposition 3.8. *We have $W(\rho_{M^0}) = \Omega(\rho_{M^0})$.*

That is, all elements of $W(\rho_{M^0})$ have length zero with respect to $\ell_{\mathcal{K}\text{-rel}}$.

Proof. For each $w \in W(\rho_{M^0})$, we fix a non-zero element $\varphi_w \in \mathcal{H}(G^0(F),(K^0,\rho^0))$ with support in K^0wK^0. To prove the proposition, it suffices to show that for any $w_1, w_2 \in W(\rho_{M^0})$, we have $\varphi_{w_1} * \varphi_{w_2} \in \mathbb{C}\cdot\varphi_{w_1w_2}$. According to Proposition 3.4, we have $W(\rho_{M^0}) = \langle s, s'\rangle \times \langle z\rangle$ or $W(\rho_{M^0}) = \langle s, s'\rangle \times \langle z, \epsilon_{M^0}\rangle$, and we can check easily that if $w_1, w_2 \in W(\rho_{M^0})$ satisfy $\ell_{\mathrm{prim}}(w_1w_2) = \ell_{\mathrm{prim}}(w_1) + \ell_{\mathrm{prim}}(w_2)$, then we have

$$\widetilde{K}^0 w_1 \widetilde{K}^0 w_2 \widetilde{K}^0 = \widetilde{K}^0 w_1 w_2 \widetilde{K}^0.$$

Since we have $\widetilde{K}^0 = \widetilde{K}_{M^0}\cdot K^0$, and the group $N(\rho_{M^0})$ normalizes the group $\widetilde{K}_{M^0}$, we also obtain that

$$\begin{aligned}
K^0 w_1 K^0 w_2 K^0 &\subseteq \widetilde{K}^0 w_1 \widetilde{K}^0 w_2 \widetilde{K}^0 \cap \mathrm{SL}_8(F)\\
&= \widetilde{K}^0 w_1 w_2 \widetilde{K}^0 \cap \mathrm{SL}_8(F)\\
&= \widetilde{K}^0 w_1 w_2 \widetilde{K}_{M^0}\cdot K^0 \cap \mathrm{SL}_8(F)\\
&= \widetilde{K}^0 \cdot \widetilde{K}_{M^0} w_1 w_2 K^0 \cap \mathrm{SL}_8(F)\\
&= \widetilde{K}^0 w_1 w_2 K^0 \cap \mathrm{SL}_8(F)\\
&= \left(\widetilde{K}^0 \cap \mathrm{SL}_8(F)\right) w_1 w_2 K^0\\
&= K^0 w_1 w_2 K^0.
\end{aligned}$$

In particular, in this case, we obtain that $\varphi_{w_1} * \varphi_{w_2} \in \mathbb{C}\cdot\varphi_{w_1w_2}$. Thus, to prove the proposition, it now suffices to show that $\varphi_s * \varphi_s \in \mathbb{C}\cdot\varphi_1$ and $\varphi_{s'} * \varphi_{s'} \in \mathbb{C}\cdot\varphi_1$. Similar calculations as above imply that $K^0sK^0sK^0 = K^0 \cup K^0sK^0$ and $K^0s'K^0s'K^0 = K^0 \cup K^0s'K^0$. Hence, we obtain that

$$\varphi_s * \varphi_s \in \mathbb{C}\cdot\varphi_s \oplus \mathbb{C}\cdot\varphi_1 \qquad \text{and} \qquad \varphi_{s'} * \varphi_{s'} \in \mathbb{C}\cdot\varphi_{s'} \oplus \mathbb{C}\cdot\varphi_1.$$

Thus, it suffices to prove that $(\varphi_s * \varphi_s)(\widetilde{s}) = (\varphi_{s'} * \varphi_{s'})(\widetilde{s}') = 0$. We take a set of representatives for $K^0/\left(K^0 \cap {}^{\widetilde{s}}K^0\right)$ as $\{u(x) \mid x \in \mathcal{O}_{E_2}/\mathfrak{p}_{E_2}\}$, where $u(x) = \left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, 1\right)$. Then, we can calculate the convolution product $(\varphi_s * \varphi_s)(\widetilde{s})$ as

$$\begin{aligned}
(\varphi_s * \varphi_s)(\widetilde{s}) &= \sum_{h \in K^0\widetilde{s}K^0/K^0} \varphi_s(h)\cdot\varphi_s(h^{-1}\widetilde{s})\\
&= \sum_{k \in K^0/(K^0\cap{}^{\widetilde{s}}K^0)} \varphi_s(k\widetilde{s})\cdot\varphi_s(\widetilde{s}^{-1}k^{-1}s)\\
&= \sum_{x\in\mathcal{O}_{E_2}/\mathfrak{p}_{E_2}} \varphi_s(u(x)\widetilde{s})\cdot\varphi_s(\widetilde{s}^{-1}u(-x)\widetilde{s}).
\end{aligned}$$

For $x \in \mathcal{O}_{E_2}$, we have

$$\widetilde{s}^{-1}u(-x)\widetilde{s} = \left(\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}, 1\right).$$

Hence, $\widetilde{s}^{-1}u(-0)\widetilde{s} \notin K^0\widetilde{s}K^0$ and for $x \in \mathcal{O}_{E_2}^\times$, we have

$$\widetilde{s}^{-1}u(-x)\widetilde{s} = \left(\begin{pmatrix}-x^{-1} & -1\\ 0 & -x\end{pmatrix}\cdot\widetilde{s}\cdot\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}, 1\right).$$

Hence, the definition of ρ^0 implies that

$$\begin{aligned}(\varphi_s * \varphi_s)(\widetilde{s}) &= \sum_{x \in \mathcal{O}_{E_2}/\mathfrak{p}_{E_2}} \varphi_s(u(x)\widetilde{s}) \cdot \varphi_s(\widetilde{s}^{-1}u(-x)\widetilde{s}) \\ &= \varphi_s(\widetilde{s})^2 \sum_{x \in \mathcal{O}_{E_2}^\times/\left(1+\mathfrak{p}_{E_2}\right)} \left(\eta \circ N_{E_2/F}\right)(-x) \\ &= \varphi_s(\widetilde{s})^2 \sum_{x \in \mathcal{O}_F^\times/(1+\mathfrak{p}_F)} \eta(x^2) = 0,\end{aligned}$$

where the last equality follows from the fact that the restriction of the character η^2 to $\mathcal{O}_F^\times$ is non-trivial. Similarly, we can prove that $(\varphi_{s'} * \varphi_{s'})(\widetilde{s}') = 0$. □

We fix a family $\mathcal{T} = \left\{T_n \in \mathrm{Hom}_{K_{M^0}}\left({}^n\rho_{M^0}, \rho_{M^0}\right)\right\}_{n \in N(\rho_{M^0})}$ as in [AFMO24a, Choice 3.10.3] and define the 2-cocycle

$$\mu^{\mathcal{T}} : W(\rho_{M^0}) \times W(\rho_{M^0}) \to \mathbb{C}^\times$$

as in [AFMO24a, Notation 3.6.1].

Corollary 3.9. *We have an isomorphism*

$$\mathcal{H}(G^0(F), (K^0, \rho^0)) \simeq \mathbb{C}[W(\rho_{M^0}), \mu^{\mathcal{T}}],$$

and the 2*-cocycle* $\mu^{\mathcal{T}}$ *is non-trivial.*

Proof. The corollary follows from [AFMO24a, Theorem 4.4.8], Corollary 3.6, and Proposition 3.8. □

References

[AFMO24a] Jeffrey D. Adler, Jessica Fintzen, Manish Mishra, and Kazuma Ohara, *Structure of Hecke algebras arising from types*, 2024, preprint available at https://arxiv.org/abs/2408.07805.

[AFMO24b] ———, *Reduction to depth zero for tame p-adic groups via Hecke algebra isomorphisms*, 2024, preprint available at https://arxiv.org/abs/2408.07801.

[BK98] Colin J. Bushnell and Philip C. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634. MR 1643417

[Fin] Jessica Fintzen, *Supercuspidal representations: construction, classification, and characters*, preprint available at `https://www.math.uni-bonn.de/people/fintzen/IHES_Fintzen.pdf`.

[Fin21] ———, *Types for tame p-adic groups*, Ann. of Math. (2) **193** (2021), no. 1, 303–346. MR 4199732

[FKS23] Jessica Fintzen, Tasho Kaletha, and Loren Spice, *A twisted Yu construction, Harish-Chandra characters, and endoscopy*, Duke Math. J. **172** (2023), no. 12, 2241–2301. MR 4654051

[GR05] David Goldberg and Alan Roche, *Hecke algebras and* SL_n*-types*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 87–131. MR 2107039

[KY17] Ju-Lee Kim and Jiu-Kang Yu, *Construction of tame types*, Representation theory, number theory, and invariant theory, Progr. Math., vol. 323, Birkhäuser/Springer, Cham, 2017, pp. 337–357. MR 3753917

[May20] Arnaud Mayeux, *Comparison of Bushnell-Kutzko and Yu's constructions of supercuspidal representations*, arXiv e-prints (2020), arXiv:2001.06259v2.

[Mor93] Lawrence Morris, *Tamely ramified intertwining algebras*, Invent. Math. **114** (1993), no. 1, 1–54. MR 1235019

[MP94] Allen Moy and Gopal Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1-3, 393–408. MR 1253198

[MP96] ———, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121. MR 1371680

[Roc02] Alan Roche, *Parabolic induction and the Bernstein decomposition*, Compositio Math. **134** (2002), no. 2, 113–133. MR 1934305

[Yu01] Jiu-Kang Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622. MR 1824988

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