The strong Atiyah conjecture for the families of locally indicable and one-relator groups

Let G be a group with a uniform bound on the orders of finite subgroups, and let $\mathrm{lcm}(G)$ be the least common multiple of these orders. In its original form, due to W. Lück and T. Schick, the strong Atiyah conjecture predicts that the L^2-Betti numbers $\beta_i^{(2)}(G,X)$ arising from a free cocompact action of G on a CW complex X belong to $\frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$. This can be reformulated and generalized in the following way. Consider the Hilbert space $\ell^2(G)$ of square summable series with orthonormal basis G and coefficients in $\mathbb{C}$. Given a subfield K of $\mathbb{C}$, we can identify every $n \times m$ matrix A over the group ring $K[G]$ with the bounded G-equivariant operator $\phi_G^A : \ell^2(G)^n \to \ell^2(G)^m$ given by right multiplication by A. In analogy to the case of linear maps between $\mathbb{C}$-vector spaces, one can define a notion of rank rk_G, a priori taking values in $\mathbb{R}_{\geq 0}$, for these operators, and consequently for matrices over $K[G]$. In this framework, the conjecture is stated as follows (the original formulation can be shown to correspond to $K = \mathbb{Q}$).

Conjecture (The strong Atiyah conjecture over K for a group G)**.** *For every matrix* $A \in Mat_{n\times m}(K[G])$, $\mathrm{rk}_G(A) \in \frac{1}{lcm(G)}\mathbb{Z}$.

For a torsion-free group, P. Linnell realized that this is equivalent to saying that a certain ring $R_{K[G]}$, in which we can embed $K[G]$, is a division ring. Thus, from an algebraic point of view, this conjecture is intimately related to the Kaplansky's zero-divisor conjecture and the Malcev problem (i.e., the problem of embeddability of a group ring without zero-divisors into a division ring).

In this talk we will introduce the strong Atiyah conjecture from this algebraic perspective and show that it holds for the families of locally indicable and one-relator groups, a result obtained as a joint work with A. Jaikin Zapirain.