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Submit the solutions in groups of two at the lecture on Tuesday, 2018-05-29

Exercise 1. For $z \in \mathbb{C}$, define the Ahlfors–Beurling kernel $k(z) = \frac{1}{z^2}$ and transform

$$Bf(z) = \text{p.v. } k * f(z).$$

(a) Show that $\widehat{k}$ coincides with a bounded function. (Hint: cut k into pieces supported in annuli $\{2^j < |z| \leq 2^{j+1}\}$)

(b) Compute $\widehat{k}$ up to a multiplicative constant.

Exercise 2. For $s \in \mathbb{R}$, we define the norm

$$\|f\|_{H^s} = \left(\int (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}$$

and for $s \in \mathbb{N}$

$$\|f\|_{W^{s,2}} = \left(\sum_{|\alpha| \leq s} \|\partial^\alpha f\|_2^2 \right)^{1/2}$$

(for $\alpha \in \mathbb{N}^d$ we denote $|\alpha| = \sum_{i=1}^d \alpha_i$). Let H^s (resp. $W^{s,2}$) be the closure of Schwartz functions in the topology of $\| \cdot \|_{H^s}$ (resp. $\| \cdot \|_{W^{s,2}}$).

(a) Show that there is a constant $C_{d,s}$ such that $C_{d,s}^{-1} \|f\|_{W^{s,2}} \leq \|f\|_{H^s} \leq C_{d,s} \|f\|_{W^{s,2}}$ whenever s is a positive integer. Hence $f \in H^s$ if and only if $f \in W^{s,2}$.

(b) Define an inner product on H^s with $s \in \mathbb{R}$ so that $\langle f, f \rangle_{H^s} = \|f\|_{H^s}^2$.

(c) Prove that H^{-s} and H^s are isomorphic but there exists $f \in H^{-s}$ with $\|f\|_{H^s} = \infty$, that is, the isomorphism is not the identity map.