S4A5 - Graduate Seminar on the local Langlands conjectures for non-quasi-split groups

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Meeting location: SemR 1.007

Meeting time: Mondays, 10ct - 12; overview talk on Tuesday, April 4, 10ct-12

Prerequisites:

- local fields (definition and basic properties)
- a basic understanding of reductive groups and smooth representations, at least at the level of Section 2 of the following notes `https://www.math.uni-bonn.de/people/fintzen/Fintzen_CDM.pdf` (Section 3 is also helpful)
- nonabelian Galois cohomology (definition and basic properties [Ser02, I.5])

Overview of the seminar: For an overview of the seminar, please attend the overview talk on Tuesday, April 4, 10ct-12 in SemR 1.007.

Outline of the talks

Throughout the outline of the talks, we let F be a local field of characteristic zero and G be a connected reductive group defined over F. We will mostly focus on the case when F is non-archimedean, but the archimedean case should also be mentioned briefly (especially for historical motivation).

Talk 1. Non-split reductive groups (April 17)

(For most of this talk, F can be an arbitrary field.) Define a quasi-split reductive group, define forms and inner forms of reductive groups. (And, of course, provide examples!) Discuss the short exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\text{based root datum of } G) \to 1.$$

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Show that any reductive group has a unique quasi-split inner form. Explain how inner forms are parametrized and thereby obtain a classification of all reductive groups over F. Define the Langlands dual group.

Main references: [Spr79], Section 4 of [KT], [Buz]

Please talk with the speaker of Talk 2.

Talk 2. The unrefined local Langlands correspondence (April 24)

Discuss the various forms of the Weil–Deligne group of F, including the motivation coming from Grothendieck's ℓ-adic monodromy theorem. Define L-parameters valued in the Weil form of the L-group of G and explain what it means for L-parameters to be admissible and tempered. Mention the Langlands classification for L-parameters. State the unrefined form of the local Langlands correspondence - Conjecture A of [Kal16a]. Mention the example of GL_n, in which case the local Langlands correspondence is a theorem due to Harris–Taylor and Henniart. While our focus is mostly on the non-archimedean case, please also say a few words about the archimedean case.

Main references: Section 1.1 of [Kal16a], Sections 5 and 6.1 of [KT].

Please talk with the speaker of Talk 1.

Talk 3. The refined local Langlands correspondence for quasisplit groups (May 15)

State Conjectures B and C of [Kal16a], filling in the necessary background on genericity. Time permitting, explain why S_ϕ° is reductive and how $\iota_{\mathfrak{w}}$ depends on $\mathfrak{w}$. Work through the example of the four-element L-packet of $\mathrm{SL}_2(F)$ in the introduction to Section 2 of [Kal16a]. State Conjecture D after first (minimally!) summarizing the needed parts of endoscopy, in particular: Harish-Chandra characters, stable conjugacy, strongly regular elements, (stable) orbital integrals, matching functions, and the fundamental lemma.

Main references: Sections 1.2 – 1.4 of [Kal16a], Sections 3.4 and 6.2 of [KT].

Please talk with the speaker of Talk 4.

Talk 4. Unramified L-packets (May 22)

In this talk, assume G is quasi-split and splits over an unramified extension. A representation of $G(F)$ is *unramified* if it has a vector fixed by a hyperspecial maximal compact subgroup, and an L-parameter is *unramified* if it factors through the norm map of the Weil–Deligne group. Recall the Satake isomorphism and use it to classify unramified representations. State the unrefined local Langlands correspondence for unramified representations. Explain how to parameterize tempered unramified L-packets [Mis12, Corollary 3].

Main references: [Bor79, II], [Car79, 4.2], [Mis12].

Please talk with the speaker of Talk 3.

Talk 5. The local Langlands correspondence for pure inner forms (June 5)

Motivate (using an insightful example) and define pure inner twists. State and explain the refined local Langlands correspondence using pure inner twists, i.e. Conjecture E of [Kal16a]. Mention the limitations of pure inner twists by giving an example of a group that cannot be reached by inner twists from a quasi-split group. Please also include a few words about the case $F = \mathbb{R}$ and its historic relevance.

Main references: Section 2, in particular Sections 2.3 – 2.4 of [Kal16a],

Please talk with the speaker of Talk 6

Talk 6. The Kottwitz set $B(G)$ (June 19)

Let F be a non-archimedean local field and assume that G is quasi-split. Discuss Kottwitz's set $B(G)$ together with its Newton map and Kottwitz map. Discuss the subset of basic elements in detail, including the functorial injection $H^1(F, G) \to B(G)_{\text{basic}}$ and the inner forms of G determined by elements of $B(G)_{\text{basic}}$. At each step, work out the example of GL_n in detail. In the second part of the talk, state Fargues's classification result for G-bundles on the Fargues–Fontaine curve and describe the geometry of Bun_G in terms of $B(G)$.

Main references: [Kot85], [RR96], [Far20], Chapter III of [FS21].

Please talk with the speaker of Talk 5

Talk 7. The local Langlands correspondence for extended pure inner forms (June 26)

Discuss the notion of an extended pure inner twist. State Conjecture F of [Kal16a] and discuss its compatibility with Conjecture E. In the second part of the talk, discuss what is known about Conjecture F. Possible topics: classical groups, the work of Fargues–Scholze, compatibilities between different constructions of local Langlands.

[This talk will be given by guest speaker Tasho Kaletha.]

Main references: Section 2.5 of [Kal16a].

Talk 8. Cohomology of Galois gerbes (July 3)

Give examples of groups for which not all inner forms are extended pure inner forms to motivate the introduction of Galois gerbes. Define the "canonical Galois gerbe" W and the cohomology set $\mathrm{H}^1(u \to W, Z \to G)$. Summarize its basic properties, especially Lemma 3.4 and Corollary 3.8 of [Kal16b]. State the Tate-Nakayama-type isomorphism for $\mathrm{H}^1(u \to$

$W, Z \to G$) and use it to compute some examples of this group. Define rigid inner twists, explain why every inner twist underlies a rigid inner form, and describe the automorphism group of a rigid inner twist (motivating the terminology "rigid"). You may focus on the case where F is non-archimedean, but make sure to summarize the differences in the theory when F is archimedean.

Main references: [Kal16b] and Section 3 of [Kal16a]

Please talk with the speaker of Talk 9.

Talk 9. The local Langlands correspondence for rigid inner forms (July 10)

State the local Langlands correspondence for rigid inner forms, i.e. Conjecture G of [Kal16a], after introducing the required objects and notation. Explain how the validity of the local Langlands conjecture for rigid inner forms follows from the validity of the conjecture for extended pure inner forms.

Main references: Section 4 of [Kal16a] and [Kal18].

Please talk with the speaker of Talk 8.

References

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