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SS 2015

Seminar on the Picardfunctor

Tuesdays 16-18h, MZ Room 0.006
Organizational meeting: Tuesday 03.02.2015, 16h(c.t.), MZ Room 1.007.

1) The Hilbert and Quot functors:
Introduce the Hilbert and Quot functors [3, 5.1.1-5.1.3]. Then discuss the Hilbert polynomial, see [3, Theorem B.7] and the stratification of the Hilbert and Quot functors [3, 5.1.4]. Give the examples in [3, 5.1.5 (1)-(5) and 5.1.6].

2) Castelnuovo-Mumford regularity:
Introduce the notion of m-regularity for coherent sheaves on $\mathbb{P}^n_k$, see [3, 5.2], and prove [3, Theorem 5.3].

3)The Flattening Stratification:
Prove the main theorem on the existence of the flattening stratification [3, Theorem 5.13].

4) Construction of the Hilbert and Quot schemes:
Prove the representability of the Quot functor in the noetherian case [3, Theorem 5.15].

5) Variants and Examples:
As an application of the construction of the Quot scheme construct the scheme of morphisms [3, Theorem 5.23] (including the exercises of loc. cit.) and the quotient by a flat projective equivalence relation [3, Theorem 5.25].

6) Grothendieck topologies:
Introduce the notion of a Grothendieck topology [2, 1.6], [3, 2.3.1] and discuss the examples of the étale topology [2, 2.1], the fppf-topology and the fpqc-topology [3, 2.3.2]. Discuss sheaves, pre-sheaves and sheafification, see [3, 2.3.3, 2.3.7] and [1, 8.1, p. 201].

7) Faithfully flat descent:
Prove [3, Lemma 2.60]. Then discuss faithfully flat descent [2, Théorème 1.4.5], [1, 6.1. Theorem 4] and prove that representable functors are sheaves for the fpqc-topology [3, Theorem 2.55], [1, 8.1. Proposition 1].

8) The various Picard functors:
Introduce the absolute and relative Picard functor [3, Definition 9.1.1, 9.1.2] and explain why the absolute Picard functor is not a separated pre-sheaf for the fppf-topology, see also [1, 8.1] Prove the comparison between the various Picard functors [3, Theorem 9.2.5] and discuss [3, Exercise 9.2.3, 9.2.4].

9) Relative effective divisors:
Introduce the notion of a relative effective divisor [3, 9.3.1] and prove that the

functor $\underline{\mathrm{Div}}_{X/S}$ is representable [3, Theorem 9.3.7], resp. part II of the proof of [1, 8.2 Theorem 5]. Introduce the module $\mathcal{Q}$ of [3, 9.3.10] and discuss the functor represented by $\mathbb{P}(\mathcal{Q})$, see [3, Theorem 9.3.13].

10) Construction of the Picard scheme:
Prove the existence of the Picard scheme for a flat, projective morphism $f : X \to S$ whose geometric fibers are reduced and irreducible [3, Theorem 9.4.8], resp. part III of the proof of [1, 8.2 Theorem 5]. If time permits discuss [3, Example 9.4.14].

11) The connected component of the identity:
Show that the connected component of the identity $\underline{\mathrm{Pic}}^0_{X/k}$ is of finite type for X a geometrically integral projective k-scheme [3, Proposition 9.5.3] and show that the tangent space of the Picard scheme $\underline{\mathrm{Pic}}_{X/k}$ can be identified with the first cohomology of the structure sheaf $H^1(X, \mathcal{O}_X)$, see [3, Theorem 9.5.11] and [1, 8.4. Theorem 1]. Deduce the dimension estimate for the Picard scheme [3, Corollary 9.5.13]. Show that $\underline{\mathrm{Pic}}^0_{X/k}$ is proper if in addition X is smooth over k. Prove Cartier's theorem [4, Theorem 1, p.167] on smoothness of group schemes in characteristic 0 and deduce that $\underline{\mathrm{Pic}}^0_{X/k}$ is an abelian variety if k is a field of characteristic 0 and X is smooth, projective and geometrically connected.

References

[1] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Modles*, Ergebnisse der Math. und ihrer Grenzgebiete, 3. Folge, Band 21, Springer.

[2] P. Deligne, *Cohomologie étale: les points de départ*, in SGA 4 1/2.

[3] B. Fantechi, L. Göttsche, L. Illusie, S. Kleiman, N. Nitsure, A. Vistoli, *Fundamental Algebraic Geometry*, Math. Surveys and Monographs **123**, American Math. Soc.

[4] D. Mumford, *Lectures on curves on an algebraic surface*, Ann. of Math. Studies **59**, Princeton University Press.